

A convergent iterative method for a logistic chemotactic system

Un método iterativo convergente para un sistema logístico quimiotáctico

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ABSTRACT. In this paper we study a nonlinear system of differential equations arising in chemotaxis. The system consists of a PDE that describes the evolution of a population and another which models the concentration of a chemical substance. In particular, we prove the existence and uniqueness of nonnegative solutions via an iterative method. First, we generate a Cauchy sequence of approximate solutions from a linear modification of the original system. Next, some uniform bounds on the solutions are used to find a subsequence that converges weakly to the solution of the original system.

Key words and phrases. reaction-diffusion equations, weak solution, convergence.

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RESUMEN. En este artículo estudiamos un sistema no lineal de ecuaciones diferenciales que aparecen en quimiotaxis. El sistema consiste de una EDP que describe la evolución de una población y otra que modela la concentración de una sustancia química. En particular, probamos la existencia y unicidad de soluciones no negativas vía un método iterativo. Primero generamos una sucesión de Cauchy de soluciones aproximadas a partir de una modificación lineal del sistema original. Luego, algunas cotas uniformes de las soluciones son usadas para encontrar una subsucesión débilmente convergente a la solución del sistema original.

Palabras y frases clave. ecuaciones de reacción-difusión, solución débil, convergencia.

1. Introduction

Chemotaxis systems have received considerable attention because they describe several biological phenomena such as leukocyte movement, self-organization during embryonic development, wound healing and cancer growth [8, 9]. These are phenomena where a population of cells moves towards a chemical signal emitted by a substance, or another population, called chemoattractant. Various forms of the system and boundary condition have been studied (cf. [5, 3, 6, 12]).

Of special interest is the following Chemotaxis system:

$$\partial_t c - D_c \Delta c = \frac{s\rho}{\beta + \rho} - \gamma c, \quad \text{in } \Omega \times (0, T), \quad (1)$$

$$\partial_t \rho - D_\rho \Delta \rho + \alpha \nabla \cdot (\rho \nabla c) = r\rho(\rho_\infty - \rho), \quad \text{in } \Omega \times (0, T), \quad (2)$$

$$\frac{\partial c}{\partial \eta} = 0, \quad \frac{\partial \rho}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times (0, T), \quad (3)$$

$$c(x, 0) = c_0(x), \quad \rho(x, 0) = \rho_0(x), \quad \text{on } \Omega. \quad (4)$$

where $\Omega \subset \mathbb{R}^N$, ($N = 1, 2, 3$) is a bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial\eta$ denotes the derivative with respect to the outer normal of $\partial\Omega$ and $T > 0$ is a fixed time.

The above problem arises from the study of pattern formation on animal coats, where pigment cells both respond to and produce their own chemoattractant [11, 10, 7]. In the biological interpretation $\rho = \rho(\mathbf{x}, t)$ and $c = c(\mathbf{x}, t)$ represent the pigment cell density and the chemoattractant concentration respectively at position \mathbf{x} and time t . The constants D_ρ and D_c are the cells and chemoattractant diffusion coefficient respectively, and α is the chemotaxis coefficient. It is assumed that cell population grows logistically where $r\rho_\infty$ is the linear mitotic growth rate with r and ρ_∞ both nonnegative constants. The chemoattractant production by the cells is given by a simple Michaelis-Menten kinetics and its consumption is linear. The constants s, β and γ are nonnegative.

Concerning to the well-posedness of the system (1)-(4) many advances have been done in the recent years [13, 2] and [1]. Specially, in [1] is proven the existence and uniqueness of classical solution for all positive values of α, ρ_∞ and r . The proof uses semigroup techniques, parabolic Schauder estimates and contraction arguments.

The aim of this paper is to get the local-in-time existence and uniqueness of a weak solution to (1)-(4) in one, two and three dimensions with proper assumptions on the initial data. Before stating our main results, we give the definition of a weak solution.

Definition 1.1. A weak solution of (1) - (4) is a pair (c, ρ) of functions satisfying the following conditions, $c(\mathbf{x}, t) \geq 0$ and $\rho(\mathbf{x}, t) \geq 0$, for a.e $(\mathbf{x}, t) \in$

$\Omega \times (0, T)$,

$$\begin{aligned} c, \rho &\in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \\ \partial_t c, \partial_t \rho &\in L^2(0, T; L^2(\Omega)) \end{aligned}$$

and for all $\phi \in H^1(\Omega)$,

$$\int_{\Omega} \partial_t c \phi \, dx + \int_{\Omega} D_c \nabla c \nabla \phi \, dx + \int_{\Omega} \gamma c \phi \, dx = \int_{\Omega} \left(\frac{s\rho}{\beta + \rho} \right) \phi \, dx, \quad (5)$$

$$\int_{\Omega} \partial_t \rho \phi \, dx + \int_{\Omega} D_\rho \nabla \rho \nabla \phi \, dx - \int_{\Omega} \alpha (\rho \nabla c) \nabla \phi \, dx = \int_{\Omega} r \rho (\rho_\infty - \rho) \phi \, dx, \quad (6)$$

a.e. in $[0, T]$.

The main result is the following existence and uniqueness theorem for weak solutions.

Theorem 1.2. *If $c_0, \rho_0 \in H^3(\Omega)$ with $0 \leq c_0$ and $0 \leq \rho_0 \leq \rho_\infty$ in Ω , then there exists $T > 0$ such that the system (1) - (4) has a unique weak solution in the sense of Definition 1.1. Furthermore, c and ρ belong to the space $L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H^3(\Omega))$.*

Our proof is based on generate a convergent sequence of approximate solutions of the nonlinear system (1)-(4). To this aim, we perform a successive substitution strategy, such that the nonlinear system (1)-(4) is replaced by a sequence of linear partial differential equations.

We start taking as initial value of the iteration the weak solutions $c^0, \rho^0 \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ of the homogeneous system

$$\begin{cases} \partial_t c^0 - D_c \Delta c^0 + \gamma c^0 = 0, & \text{in } \Omega \times (0, T), \\ \frac{\partial c^0}{\partial \eta} = 0, & \text{on } \partial\Omega \times (0, T), \quad c^0(x, 0) = c_0(x), \text{ for } x \in \Omega. \end{cases} \quad (7)$$

$$\begin{cases} \partial_t \rho^0 - D_\rho \Delta \rho^0 + \alpha \nabla \cdot (\rho^0 \nabla c^0) = 0 & \text{in } \Omega \times (0, T) \\ \frac{\partial \rho^0}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0, T), \quad \rho^0(x, 0) = \rho_0(x) \text{ for } x \in \Omega. \end{cases} \quad (8)$$

In addition, for $k \in \mathbb{N}_0$ let $c^{k+1}, \rho^{k+1} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ be the weak solutions to the nonhomogeneous system

$$\begin{cases} \partial_t c^{k+1} - D_c \Delta c^{k+1} + \gamma c^{k+1} = \frac{s\rho^k}{\beta + \rho^k}, & \text{in } \Omega \times (0, T), \\ \frac{\partial c^{k+1}}{\partial \eta} = 0 & \text{on } \partial\Omega \times (0, T), \quad c^{k+1}(x, 0) = c_0(x) \text{ for } x \in \Omega, \end{cases} \quad (9)$$

$$\begin{cases} \partial_t \rho^{k+1} - D_\rho \Delta \rho^{k+1} + \alpha \nabla \cdot (\rho^{k+1} \nabla c^{k+1}) = r \rho^k (\rho_\infty - \rho^k), & \text{in } \Omega \times (0, T), \\ \frac{\partial \rho^{k+1}}{\partial \eta} = 0, & \text{on } \partial\Omega \times (0, T), \quad \rho^{k+1}(x, 0) = \rho_0(x) \text{ for } x \in \Omega. \end{cases} \tag{10}$$

To prove theorem 1.2 we first prove existence and uniqueness of weak solutions to the homogeneous problems (7) and (8) by applying the standard theory for linear PDE. These solutions c^0 and ρ^0 are sufficient regular, that the standard theory for linear PDE guarantee the existence and uniqueness of the successive iterates (c^k, ρ^k) $k = 1, 2, \dots$. Next, we show that the generated solutions sequence is a bounded Cauchy sequence, and its limit is the solution of (1)-(4).

2. Detail of Proof

Lemma 2.1. (Properties of iterative Sequence). *Under the assumptions of theorem 1.2, there exists $T > 0$ such that:*

(i) *There exists a unique weak solution to the system (7)-(8) and (9)-(10) with conditions (3) and (4) and for every $k \in \mathbb{N}_0$ it holds that*

$$c^k, \rho^k \in L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H^3(\Omega)), \tag{11}$$

$$\partial_t c^k, \partial_t \rho^k \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)). \tag{12}$$

For adequate constants $\mathcal{C}(\Omega, T)$ the following estimates are satisfied

$$\begin{aligned} \text{ess sup}_{t \in [0, T]} \left[\|\partial_t c^k\|_{H^1(\Omega)} + \|c^k\|_{H^3(\Omega)} \right] + \|c^k\|_{L^2(0, T; H^4(\Omega))} + \|\partial_t c^k\|_{L^2(0, T; H^2(\Omega))} \\ \leq \mathcal{C}(\Omega, T) \left[\|c_0\|_{H^3(\Omega)} + \|f\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t f\|_{L^2(0, T; L^2(\Omega))} \right], \end{aligned} \tag{13}$$

$$\begin{aligned} \text{ess sup}_{t \in [0, T]} \left[\|\partial_t \rho^k\|_{H^1(\Omega)} + \|\rho^k\|_{H^3(\Omega)} \right] + \|\rho^k\|_{L^2(0, T; H^4(\Omega))} + \|\partial_t \rho^k\|_{L^2(0, T; H^2(\Omega))} \\ \leq \mathcal{C}(\Omega, T) \left[\|\rho_0\|_{H^3(\Omega)} + \|g\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t g\|_{L^2(0, T; L^2(\Omega))} \right]. \end{aligned} \tag{14}$$

(ii) *The functions ρ^k, c^k satisfy for all $k \in \mathbb{N}_0$, the following inequalities*

$$0 \leq c^k(\mathbf{x}, t), \quad 0 \leq \rho^k(\mathbf{x}, t) \leq \rho_\infty \text{ for a.e } \mathbf{x} \in \Omega, t \in (0, T) \tag{15}$$

Proof. The proof is by induction on k .

Verification for $k=0$: We prove, that the lemma holds for the system (7)-(8). If we write $c^0(\mathbf{x}, t) = u(\mathbf{x}, t)e^{-\gamma t}$, then

$$(\partial_t u - \gamma u)e^{-\gamma t} = D_c e^{-\gamma t} \Delta u - \gamma u e^{-\gamma t}$$

which simplifies to

$$\partial_t u = D_c \Delta u.$$

Hence, $c^0(\mathbf{x}, t)$ equals some solution $u(\mathbf{x}, t)$ of the diffusion equation, multiplied by an exponentially decay term. Since $c_0 \in H^3(\Omega)$ and the compatibility conditions are fulfilled trivially, the regularity theory of linear parabolic equations [4] implies that

$$c^0 \in L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H^3(\Omega)) \quad (16)$$

$$c_t^0 \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \quad (17)$$

and

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} \left[\|c_t^0\|_{H^1(\Omega)} + \|c^0\|_{H^3(\Omega)} \right] + \|c^0\|_{L^2(0, T; H^4(\Omega))} \\ + \|c_t^0\|_{L^2(0, T; H^2(\Omega))} \leq \mathcal{C}(\Omega, T) \|c_0\|_{H^3(\Omega)}. \end{aligned} \quad (18)$$

That $c^0(\mathbf{x}, t) \geq 0$ a.e in $\Omega \times (0, T)$ follows from the maximum principle for the diffusion equation.

To prove that ρ^0 satisfies the lemma, we start writing the equation (8) as follows

$$\frac{\partial \rho^0}{\partial t} - D_\rho \Delta \rho^0 + \alpha \nabla c^0 \cdot \nabla \rho^0 + \alpha \Delta c^0 \rho^0 = 0. \quad (19)$$

Since c^0 is known, equation (19) is linear. To show existence and uniqueness of $\rho^0 \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ it is sufficient to see that

(a) The coefficients $\alpha \nabla c^0$ and $\alpha \Delta c^0$ belong to $L^\infty(\Omega_T)$.

and

(b) There exist some $\mu > 0$ and $\kappa \geq 0$ such that for all $0 \leq t \leq T$

$$\mu \|\rho^0\|_{H^1(\Omega)}^2 \leq B[\rho^0, \rho^0; t] + \kappa \|\rho^0\|_{L^2(\Omega)}^2. \quad (20)$$

Where $B[\rho, v, t]$ denotes the bilinear form

$$B[\rho, v; t] := \int_{\Omega} \left(D_\rho \nabla \rho^0 \cdot \nabla v + \alpha \nabla c^0 \cdot \nabla \rho^0 v + \alpha \Delta c^0 \rho^0 v \right) d\mathbf{x}. \quad (21)$$

for $\rho, v \in H^1(\Omega)$, a.e. $0 \leq t \leq T$.

Item (a) follows from the fact that $c^0 \in L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H^3(\Omega))$ and the Sobolev embedding of $H^2(\Omega)$ in $C(\bar{\Omega})$ for Ω open subset of \mathbb{R}^N , $N = 1, 2, 3$.

In order to prove (b), first observe that by the uniformly elliptic property, there exists a constant $\theta > 0$ such that

$$\int_{\Omega} D_\rho \nabla \rho^0 \cdot \nabla \rho^0 \geq \theta \|\nabla \rho^0\|_{L^2(\Omega)}^2.$$

Furthermore, for all $\varepsilon > 0$,

$$\begin{aligned} \int_{\Omega} (\nabla c^0 \cdot \nabla \rho^0) \rho^0 \, d\mathbf{x} &\geq -\|\nabla c^0\|_{L^\infty(\Omega_T)} \|\nabla \rho^0\|_{L^2(\Omega)} \|\rho^0\|_{L^2(\Omega)} \\ &\geq -\frac{1}{2} \|\nabla c^0\|_{L^\infty(\Omega)} \left[\varepsilon \|\nabla \rho^0\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|\rho^0\|_{L^2(\Omega)}^2 \right] \end{aligned} \quad (22)$$

and

$$\int_{\Omega} \Delta c^0 (\rho^0)^2 \, d\mathbf{x} \geq -\|\Delta c^0\|_{L^\infty(\Omega_T)} \|\rho^0\|_{L^2(\Omega)}^2.$$

Thus, for all $\varepsilon \in (0, 2\theta/\|\nabla c^0\|_{L^\infty(\Omega_T)})$, the inequality (20) holds, with

$$\kappa = \|\Delta c^0\|_{L^\infty(\Omega_T)} + \frac{1}{2\varepsilon} \|\nabla c^0\|_{L^\infty(\Omega_T)} \quad \text{and} \quad \mu = \theta - \frac{\varepsilon}{2} \|\nabla c^0\|_{L^\infty(\Omega_T)}. \quad (23)$$

Applying, the theory of linear parabolic equations in [4], we get the existence and uniqueness of the weak solution ρ^0 . In addition, since the initial data ρ_0 is in $H^3(\Omega)$, theorem 7.16 in [4] implies that ρ^0 satisfies (11), (12) and (14).

The task is now to show that $\rho^0 \geq 0$. We test with $(\rho^0)^- := \min(\rho^0, 0)$ the variational formulation of (7), then

$$\frac{d}{dt}(\rho^0, (\rho^0)^-) + B[\rho^0, (\rho^0)^-; t] = 0. \quad (24)$$

After adding $\kappa \|(\rho^0)^-\|_{L^2(\Omega)}^2$ to both sides of (24), and applying property (20), we get

$$\frac{d}{dt} \|(\rho^0)^-\|_{L^2(\Omega)}^2 \leq \kappa \|(\rho^0)^-\|_{L^2(\Omega)}^2. \quad (25)$$

By Gronwall's lemma, we can now deduce that

$$\|(\rho^0)^-(t)\|_{L^2(\Omega)}^2 \leq \|(\rho^0)^-(0)\|_{L^2(\Omega)}^2 = 0 \quad (26)$$

since $(\rho^0)^-(0) = \rho_0 \geq 0$ by assumption. Then $(\rho^0)^-(t) = 0$ almost everywhere in $\Omega \times (0, T)$, and therefore $\rho^0 \geq 0$ almost everywhere in $\Omega \times (0, T)$.

To show the upper bound of ρ^0 , we use the same trick but test now with $(\rho^0 - \rho_\infty)^+ := \max(\rho^0 - \rho_\infty, 0)$. As ρ_∞ is a constant we have $\partial_t \rho_\infty = \nabla \rho_\infty = \Delta \rho_\infty = 0$ and therefore

$$\frac{d}{dt}(\rho^0, (\rho^0 - \rho_\infty)^+) + B[\rho^0, (\rho^0 - \rho_\infty)^+; t] = 0$$

is equivalent to

$$\frac{1}{2} \frac{d}{dt} \|(\rho^0 - \rho_\infty)^+\|_{L^2(\Omega)}^2 + B[(\rho^0 - \rho_\infty)^+, (\rho^0 - \rho_\infty)^+; t] = 0.$$

Property (20) of B implies

$$\frac{d}{dt} \|(\rho^0 - \rho_\infty)^+\|_{L^2(\Omega)}^2 \leq \kappa \|(\rho^0 - \rho_\infty)^+\|_{L^2(\Omega)}^2.$$

Now, we use Gronwall's lemma and the fact that $\rho_0 \leq \rho_\infty$ to deduce

$$\|(\rho^0 - \rho_\infty)^+\|_{L^2(\Omega)}^2 \leq \|(\rho^0(0) - \rho_\infty)^+\|_{L^2(\Omega)}^2 = 0.$$

Therefore $(\rho^0 - \rho_\infty)^+ = 0$ almost everywhere in $\Omega \times (0, T)$, which yields $\rho^0 \leq \rho_\infty$ almost everywhere in $\Omega \times (0, T)$.

Induction hypothesis: Assume the lemma holds for k .

Induction step ($k \rightarrow k+1$): By induction hypothesis $0 \leq \rho^k(\mathbf{x}, t) \leq \rho_\infty$ for a.e $\mathbf{x} \in \Omega$, $t \in [0, T]$, then it is easy to see that the right hand sides

$$f(\rho^k(\mathbf{x}, t)) := \frac{s\rho^k}{\beta + \rho^k} \quad \text{and} \quad g(\rho^k(\mathbf{x}, t)) := r\rho^k(\rho_\infty - \rho^k) \quad (27)$$

of equations (9) and (10) belong to the space $L^2(0, T; L^2(\Omega))$. Indeed

$$\int_0^T \|f(\rho^k)\|_{L^2(\Omega)}^2 dt = \int_0^T \left\| \frac{s\rho^k}{\beta + \rho^k} \right\|_{L^2(\Omega)}^2 dt \quad (28)$$

$$= \int_0^T \int_\Omega \left(\frac{s\rho^k}{\beta + \rho^k} \right)^2 dx dt \quad (29)$$

$$\leq \int_0^T \int_\Omega s^2 dx dt \quad (30)$$

$$\leq s^2 |\Omega| T \quad (31)$$

and

$$\int_0^T \|g(\rho^k)\|_{L^2(\Omega)}^2 dt = \int_0^T \left\| r\rho^k(\rho_\infty - \rho^k) \right\|_{L^2(\Omega)}^2 dt \quad (32)$$

$$= \int_0^T \int_\Omega (r\rho^k(\rho_\infty - \rho^k))^2 dx dt \quad (33)$$

$$\leq \int_0^T \int_\Omega \frac{r^2 \rho_\infty^4}{16} dx dt \quad (34)$$

$$\leq \frac{r^2 \rho_\infty^4}{16} |\Omega| T. \quad (35)$$

Now the linear theory yields the existence of a unique weak solution of (9) and (10) with initial data (4) and boundary conditions (3). The solution (c^{k+1}, ρ^{k+1}) satisfies

$$c^{k+1}, \rho^{k+1} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

In order to see that c^{k+1} and ρ^{k+1} satisfy the regularity properties (13) and estimate (14), we apply theorem 7.1.6 in [4]. Then, it is sufficient to prove that $f(\rho^k)$ and $g(\rho^k)$ belongs to the space $L^2(0, T; H^2(\Omega))$ and $\partial_t f(\rho^k), \partial_t g(\rho^k) \in L^2(0, T; L^2(\Omega))$. To this end, we observe that:

- The functions $f(x)$ and $g(x)$ in (27) are continuous differentiable functions for all $x \in \mathbb{R}_+$.
- By induction hypothesis c^k and ρ^k belong to $H^4(\Omega)$ and the Sobolev embedding $H^4(\Omega) \subset C^2(\bar{\Omega})$, we have that c^k and ρ^k are $C^2(\bar{\Omega})$ functions. Further, $\rho_\infty \geq \rho^k \geq 0$ almost everywhere in $\Omega \times (0, T)$.

Hence $f(\rho^k(\mathbf{x}, t))$ and $g(\rho^k(\mathbf{x}, t))$ belong to $H^2(\Omega)$ a.e. $t \in [0, T]$ and

$$\int_0^T \|f(\rho^k)\|_{H^2(\Omega)}^2 dt < \infty \quad \text{and} \quad \int_0^T \|g(\rho^k)\|_{H^2(\Omega)}^2 dt < \infty$$

i.e., $f(\rho^k(\mathbf{x}, t)), g(\rho^k(\mathbf{x}, t)) \in L^2(0, T; H^2(\Omega))$.

In addition, $\partial_t f(\rho^k) \in L^2(0, T; L^2(\Omega))$ since

$$\int_0^T \|\partial_t f(\rho^k)\|_{L^2(\Omega)}^2 dt = \int_0^T \left\| \frac{s\beta}{(\beta + \rho^k)^2} \partial_t \rho^k \right\|_{L^2(\Omega)}^2 dt \quad (36)$$

$$= \int_0^T \int_\Omega \left(\frac{s\beta}{(\beta + \rho^k)^2} \partial_t \rho^k \right)^2 dx dt \quad (37)$$

$$\leq \int_0^T \left(\frac{s}{\beta} \right)^2 \int_\Omega (\partial_t \rho^k)^2 dx dt \quad (38)$$

$$= \int_0^T \left(\frac{s}{\beta} \right)^2 \|\partial_t \rho^k\|_{L^2(\Omega)}^2 dt \quad (39)$$

$$= \left(\frac{s}{\beta} \right)^2 \|\partial_t \rho^k\|_{L^2(0, T; L^2(\Omega))}^2 \quad (40)$$

$$\leq \mathcal{C}(\Omega, T) \|\rho_0\|_{H^3(\Omega)}^2. \quad (41)$$

We next show that $\partial_t g(\rho^k)$ belongs to $L^2(0, T; L^2(\Omega))$:

$$\int_0^T \|\partial_t g(\rho^k)\|_{L^2(\Omega)}^2 dt = \int_0^T \left\| r(\rho_\infty - 2\rho^k) \partial_t \rho^k \right\|_{L^2(\Omega)}^2 dt \quad (42)$$

$$= \int_0^T \int_\Omega \left(r(\rho_\infty - 2\rho^k) \partial_t \rho^k \right)^2 dx dt \quad (43)$$

$$(44)$$

$$\leq \int_0^T (r\rho_\infty)^2 \int_\Omega (\partial_t \rho^k)^2 dx dt \quad (45)$$

$$= \int_0^T (r\rho_\infty)^2 \|\partial_t \rho^k\|_{L^2(\Omega)}^2 dt \quad (46)$$

$$= (r\rho_\infty)^2 \|\partial_t \rho^k\|_{L(0,T;L^2(\Omega))}^2 \quad (47)$$

$$\leq \mathcal{C}(\Omega, T) \|\rho_0\|_{H^3(\Omega)}^2. \quad (48)$$

We now turn to show that $c^{k+1}(\mathbf{x}, t) \geq 0$. Consider the weak formulation of (9) and test with $(c^{k+1})^- := \min(c^{k+1}, 0)$, then

$$\begin{aligned} \int_\Omega \partial_t c^{k+1} (c^{k+1})^- dx + \int_\Omega D_c \nabla c^{k+1} \nabla (c^{k+1})^- dx + \int_\Omega \gamma c^{k+1} (c^{k+1})^- dx \\ = \int_\Omega \left(\frac{s\rho^k}{\beta + \rho^k} \right) (c^{k+1})^- dx. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_\Omega |(c^{k+1})^-|^2 dx + D_c \int_\Omega |\nabla (c^{k+1})^-|^2 dx + \gamma \int_\Omega |(c^{k+1})^-|^2 dx \\ = \int_\Omega \left(\frac{s\rho^k}{\beta + \rho^k} \right) (c^{k+1})^- dx. \end{aligned}$$

which gives by integration in time

$$\begin{aligned} \frac{1}{2} \int_\Omega |(c^{k+1})^-(t)|^2 dx + D_c \int_0^t \int_\Omega |\nabla (c^{k+1})^-|^2 dx ds + \gamma \int_0^t \int_\Omega |(c^{k+1})^-|^2 dx ds \\ = \int_\Omega \int_0^t \left(\frac{s\rho^k}{\beta + \rho^k} \right) (c^{k+1})^- dx ds + \frac{1}{2} \int_\Omega |(c^{k+1})^-(0)|^2 dx. \end{aligned}$$

As $(c^{k+1})^-(0) = (c_0)^- = 0$ and $\rho^k \geq 0$ by induction hypothesis, we deduce

$$\frac{1}{2} \int_\Omega |(c^{k+1})^-(t)|^2 dx + D_c \int_0^t \int_\Omega |\nabla (c^{k+1})^-|^2 dx ds + \gamma \int_0^t \int_\Omega |(c^{k+1})^-|^2 dx ds \leq 0,$$

that is to say that $(c^{k+1})^- = 0$ a.e in $(0, T) \times \Omega$ and therefore $c^{k+1} \geq 0$ a.e in $\Omega \times (0, T)$.

Remark 2.2. If $\gamma \geq 1$ then $c^{k+1}(\mathbf{x}, t) \leq S$ a.e in $\Omega \times (0, T)$.

It remains to show that $0 \leq \rho^{k+1} \leq \rho_\infty$. For the positivity of ρ^{k+1} , we use the variational formulation of (10) and test with $(\rho^{k+1})^- := \min(\rho^{k+1}, 0)$, this yields

$$\frac{d}{dt} (\rho^{k+1}, (\rho^{k+1})^-) + B[\rho^{k+1}, (\rho^{k+1})^-; t] = (r\rho^k(\rho_\infty - \rho^k), (\rho^{k+1})^-). \quad (49)$$

By induction hypothesis $0 \leq \rho^k \leq \rho_\infty$, then from (49) we get that

$$\frac{1}{2} \frac{d}{dt} \|(\rho^{k+1})^-\|_{L^2(\Omega)}^2 + B[(\rho^{k+1})^-, (\rho^{k+1})^-; t] \leq 0.$$

Adding to both sides $\kappa \|(\rho^{k+1})^-\|_{L^2(\Omega)}^2$ with κ as in (23), we obtain

$$\frac{1}{2} \frac{d}{dt} \|(\rho^{k+1})^-\|_{L^2(\Omega)}^2 \leq \kappa \|(\rho^{k+1})^-\|_{L^2(\Omega)}^2$$

and applying Gronwall's lemma, we can deduce that

$$\frac{1}{2} \|(\rho^{k+1}(t))^-\|_{L^2(\Omega)}^2 \leq \|(\rho^{k+1}(0))^-\|_{L^2(\Omega)}^2 e^{\kappa t} = 0$$

since $\rho^{k+1}(0) = \rho_0 \geq 0$ by assumption. This yields $(\rho^{k+1}(t))^- = 0$ a.e in $\Omega \times (0, T)$ and therefore $\rho^{k+1} \geq 0$ a.e in $\Omega \times (0, T)$.

Finally, we have to show that ρ^{k+1} is bounded from above by ρ_∞ a.e on $\Omega \times (0, T)$. Testing the variational formulation of (10) with $(\rho^{k+1} - \rho_\infty)^+$, we find by the rules of calculus Sobolev spaces that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2 + B[(\rho^{k+1} - \rho_\infty)^+, (\rho^{k+1} - \rho_\infty)^+; t] \\ = (r\rho^k(\rho_\infty - \rho^k), (\rho^{k+1} - \rho_\infty)^+). \end{aligned} \quad (50)$$

After adding $\kappa \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2$ to both sides of (50), taking in account the inequality (20) and that ρ^k is bounded, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2 &\leq \kappa \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2 + (r\rho^k(\rho_\infty - \rho^k), (\rho^{k+1} - \rho_\infty)^+) \\ \frac{d}{dt} \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2 &\leq 2\kappa \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2 + r^2 \frac{\rho_\infty^4}{8} \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2 \\ &\leq (2\kappa + r^2 \frac{\rho_\infty^4}{8}) \|(\rho^{k+1} - \rho_\infty)^+\|_{L^2(\Omega)}^2. \end{aligned} \quad (51)$$

Gronwall's inequality and the fact that $\rho^{k+1}(0) \leq \rho_\infty$ imply

$$\|(\rho^{k+1}(t) - \rho_\infty)^+\|_{L^2(\Omega)}^2 \leq \|(\rho^{k+1}(0) - \rho_\infty)^+\|_{L^2(\Omega)}^2 e^{\int_0^t 2\kappa + r^2 \frac{\rho_\infty^4}{8} ds} = 0.$$

Thus $\rho^{k+1} \leq \rho_\infty$ a.e in $\Omega \times (0, T)$.

This completes the induction proof. \square

Proof. of Theorem 1.2

Existence: We show that the iterative sequence constructed above is a Cauchy sequence, which will lead to the existence of the solution (c, ρ) as its limit.

Let $k \in \mathbb{N}$ be arbitrary. Since c^k and c^{k+1} solve (9) with the same initial data and $c^k, c^{k+1} \in L^2(0, T, H^2(\Omega)) \cap L^\infty(0, T, H^1(\Omega))$, (by Lemma 2.1), then theorem 7.1.5 in [4] implies

$$\begin{aligned} & \|c^{k+1} - c^k\|_{L^\infty(0, T, H^1(\Omega))}^2 + \|c^{k+1} - c^k\|_{L^2(0, T, H^2(\Omega))}^2 \\ & \leq C(\Omega, T) \|f(\rho^k) - f(\rho^{k-1})\|_{L^2(0, T, L^2(\Omega))}^2 \\ & = C(\Omega, T) \int_0^T \left\| \frac{s\rho^k}{\beta + \rho^k} - \frac{s\rho^{k-1}}{\beta + \rho^{k-1}} \right\|_{L^2(\Omega)}^2 dt \\ & = C(\Omega, T) \int_0^T \left\| \frac{s\beta(\rho^k - \rho^{k-1})}{(\beta + \rho^k)(\beta + \rho^{k-1})} \right\|_{L^2(\Omega)}^2 dt \\ & \leq C(\Omega, T) \left(\frac{s}{\beta}\right)^2 \int_0^T \|\rho^k - \rho^{k-1}\|_{L^2(\Omega)}^2 dt \quad (52) \\ & \leq C(\Omega, T) \left(\frac{s}{\beta}\right)^2 T \|\rho^k - \rho^{k-1}\|_{L^\infty(0, T, H^1(\Omega))}^2 \quad (53) \end{aligned}$$

for $0 < T \leq T_1$ with $T_1 = \min\{\frac{1}{8}, \frac{\beta^2}{C(\Omega, T)s^2}\}$.

Similarly, due to (10) and theorem 7.1.5 in [4], we estimate

$$\begin{aligned} & \|\rho^{k+1} - \rho^k\|_{L^\infty(0, T, H^1(\Omega))}^2 \\ & \leq C(\Omega, T) \|\nabla(c^{k+1} - c^k)\nabla\rho^k + \Delta(c^{k+1} - c^k)\rho^k + g(\rho^k) - g(\rho^{k-1})\|_{L^2(0, T, L^2(\Omega))}^2 \\ & \leq 3C(\Omega, T) \int_0^T \left\{ \|\nabla(c^{k+1} - c^k)\nabla\rho^k\|_{L^2(\Omega)}^2 + \|\Delta(c^{k+1} - c^k)\rho^k\|_{L^2(\Omega)}^2 \right. \\ & \quad \left. + \|g(\rho^k) - g(\rho^{k-1})\|_{L^2(\Omega)}^2 \right\} dt \\ & \leq 3C(\Omega, T) \int_0^T \|\nabla(c^{k+1} - c^k)\nabla\rho^k\|_{L^2(\Omega)}^2 dt + 3C(\Omega, T) \int_0^T \|\Delta(c^{k+1} - c^k)\rho^k\|_{L^2(\Omega)}^2 dt \\ & \quad + 3C(\Omega, T) \int_0^T \|g(\rho^k) - g(\rho^{k-1})\|_{L^2(\Omega)}^2 dt \\ & = I_1 + I_2 + I_3. \quad (54) \end{aligned}$$

Now we estimate each of the three terms separately

As $n \leq 3$, by the Sobolev embedding there exist $C_1 > 0$ such that

$$\|w\|_{L^4(\Omega)} \leq C_1 \|w\|_{H^1(\Omega)} \quad (55)$$

for all $w \in H^1(\Omega)$. Then

$$I_1 \leq 3 C(\Omega, T) \int_0^T \int_{\Omega} (\nabla(c^{k+1} - c^k))^2 (\nabla\rho^k)^2 dx dt \tag{56}$$

$$\leq 3 C(\Omega, T) \int_0^T \left(\int_{\Omega} (\nabla(c^{k+1} - c^k))^4 dx \right)^{1/2} \left(\int_{\Omega} (\nabla\rho^k)^4 dx \right)^{1/2} dt \tag{57}$$

$$\leq 3 C(\Omega, T) \int_0^T \|\nabla(c^{k+1} - c^k)\|_{L^4(\Omega)}^2 \|\nabla\rho^k\|_{L^4(\Omega)}^2 dt$$

$$\stackrel{(55)}{\leq} 3 C(\Omega, T)^2 C_1^4 \int_0^T \|\nabla(c^{k+1} - c^k)\|_{H^1(\Omega)}^2 \|\nabla\rho^k\|_{H^1(\Omega)}^2 dt$$

$$\stackrel{(14)}{\leq} 3 C(\Omega, T)^2 C_1^4 \|\rho_0\|_{H^3(\Omega)}^2 \int_0^T \|\nabla(c^{k+1} - c^k)\|_{H^1(\Omega)}^2 dt$$

$$\leq 3 C(\Omega, T)^2 C_1^4 \|\rho_0\|_{H^3(\Omega)}^2 \int_0^T \|c^{k+1} - c^k\|_{H^2(\Omega)}^2 dt$$

$$\stackrel{(53)}{\leq} 3 C(\Omega, T)^4 C_1^4 \|\rho_0\|_{H^3(\Omega)}^2 \frac{s^2}{\beta^2} T \|\rho^k - \rho^{k-1}\|_{L^\infty(0,T;H^1(\Omega))}^2 \tag{58}$$

for $0 < T \leq T_2$ with $T_2 = \min\{T_1, \frac{\beta^2}{3C(\Omega,T)^4 c_1^4 s^2 \|\rho_0\|_{H^3(\Omega)}^2}\}$.

Further, for $0 < T \leq T_3$ with $T_3 = \min\{T_1, \frac{\beta^2}{3C(\Omega,T)^2 s^2 \rho_\infty^2}\}$ we have

$$\begin{aligned} I_2 &\leq 3 C(\Omega, T) \int_0^T \|\Delta(c^{k+1} - c^k)\|_{L^2(\Omega)}^2 \|\rho^k\|_{L^\infty(\Omega)}^2 dt \\ &\leq 3 C(\Omega, T) \rho_\infty^2 \int_0^T \|c^{k+1} - c^k\|_{H^2(\Omega)}^2 dt \\ &\stackrel{(53)}{\leq} 3 C(\Omega, T)^2 \rho_\infty^2 \frac{s^2}{\beta^2} T_1 \|\rho^k - \rho^{k-1}\|_{L^\infty(0,T;H^1(\Omega))}^2, \end{aligned} \tag{59}$$

and

$$\begin{aligned} I_3 &= 3 C(\Omega, T) \int_0^T \|r\rho^k(\rho_\infty - \rho^k) - r\rho^{k-1}(\rho_\infty - \rho^{k-1})\|_{L^2(\Omega)}^2 dt \\ &= 3 C(\Omega, T) \int_0^T \|r\rho_\infty(\rho^k - \rho^{k-1})(1 - \frac{1}{\rho_\infty}(\rho^k + \rho^{k-1}))\|_{L^2(\Omega)}^2 dt \\ &= 3 C(\Omega, T) \int_0^T \int_{\Omega} |r^2 \rho_\infty^2 (\rho^k - \rho^{k-1})^2 (1 - \frac{1}{\rho_\infty}(\rho^k + \rho^{k-1}))^2| dx dt \end{aligned} \tag{60}$$

$$\begin{aligned}
&\leq 3 C(\Omega, T) \int_0^T \int_{\Omega} |r^2 \rho_{\infty}^2 (\rho^k - \rho^{k-1})^2 (1 + \frac{1}{\rho_{\infty}^2} (\rho^k + \rho^{k-1}))^2| dx dt \\
&\leq 15 C(\Omega, T) r^2 \rho_{\infty}^2 \int_0^T \int_{\Omega} |(\rho^k - \rho^{k-1})^2| dx dt \\
&\leq 15 C(\Omega, T) r^2 \rho_{\infty}^2 \int_0^T \|(\rho^k - \rho^{k-1})\|_{L^2(\Omega)}^2 dt \\
&\leq 15 C(\Omega, T) r^2 \rho_{\infty}^2 \int_0^T \|(\rho^k - \rho^{k-1})\|_{H^1(\Omega)}^2 dt \\
&\leq 15 C(\Omega, T) r^2 \rho_{\infty}^2 T \|(\rho^k - \rho^{k-1})\|_{L^{\infty}(0, T; H^1(\Omega))}^2
\end{aligned} \tag{61}$$

for $0 < T \leq T_4$ with $T_4 = \min\{T_3, \frac{1}{15C(\Omega, T)^2 r^2 \rho_{\infty}^2}\}$.

Altogether, (53), (58), (59) and (61) yield

$$\|c^{k+1} - c^k\|_{L^{\infty}(0, T; H^1(\Omega))}^2 + \|\rho^{k+1} - \rho^k\|_{L^{\infty}(0, T; H^1(\Omega))}^2 \leq \frac{1}{2} \|\rho^k - \rho^{k-1}\|_{L^{\infty}(0, T; H^1(\Omega))}^2 \tag{62}$$

whenever $0 < T \leq T_4$. That is, for $T := T_4$ the sequences $\{c^k\}$ and $\{\rho^k\}$ are Cauchy sequences in $L^{\infty}(0, T; H^1(\Omega))$ and there are functions c and ρ in $L^{\infty}(0, T; H^1(\Omega))$ such that

$$c^k \rightarrow c \text{ and } \rho^k \rightarrow \rho \text{ strongly in } L^{\infty}(0, T; H^1(\Omega)).$$

Since $L^2(0, T; H^4(\Omega))$ and $L^2(0, T; H^2(\Omega))$ are Hilbert spaces, the uniform bounds (13) and (14) imply that for subsequences c^{k_l} and ρ^{k_l}

$$\begin{aligned}
c^{k_l} &\rightharpoonup c; & \rho^{k_l} &\rightharpoonup \rho \text{ weakly in } L^2(0, T; H^4(\Omega)); \\
\partial_t c^{k_l} &\rightharpoonup c; & \partial_t \rho^{k_l} &\rightharpoonup \rho \text{ weakly in } L^2(0, T; H^2(\Omega)).
\end{aligned}$$

Using all these convergences in the weak formulation of (9), (10) and letting $l \rightarrow \infty$, we conclude that (c, ρ) is a weak solution to (1)-(4) and also satisfies (11)-(15).

Uniqueness if (c_1, ρ_1) and (c_2, ρ_2) are two weak solutions of (1)-(4), they satisfy (62). Then

$$\|c_1 - c_2\|_{L^{\infty}(0, T; H^1(\Omega))}^2 + \|\rho_1 - \rho_2\|_{L^{\infty}(0, T; H^1(\Omega))}^2 \leq \frac{1}{2} \|\rho_1 - \rho_2\|_{L^{\infty}(0, T; H^1(\Omega))}^2$$

for $T := T_4$. Therefore, both solutions coincide.

Hence, the proof of theorem 1.2 is complete. \square

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