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On the graph on a Weyl group being an interval graph

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Abstract. We consider the graph on a Weyl group whose associated root system is arbitrary. It is shown that such a graph is an interval graph only when the associated root systems are of some particular types.

Key Words and Phrases: root systems, Weyl groups, interval graph.
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1. **Introduction**

The graphs on Weyl groups were defined and studied in [1]. The basic idea was to define the graph on Weyl groups using a relation on the Weyl groups introduced in [6]. The relation on Weyl groups arose due to the technique used in proving the Verma's conjecture on Weyl's dimension polynomial. This new relation on Weyl groups gives rise to a partial order in a very natural manner. This partial order or the incidence matrix of our graph on Weyl groups has applications in the representation of algebraic Chevally groups [7]. Several problems on the graph on Weyl groups have been solved: [2], [3], [4], [5]. In this paper we determine the Weyl groups for which the associated graph is an interval graph. Here we take the root system to be arbitrary and show that the graph on the Weyl group with such a root system is an interval graph only when the root system is essentially of the type A_2, A_3, B_2, $A_3 \times A_3$, $A_3 \times B_2$ or

$B_2 \times B_2$. We summarize below few facts about root systems and Weyl groups but for details we refer to [8].

Let E be a Euclidean space of dimension n with a positive definite inner product $(\ ,\ )$. For any vector $\alpha \in E$ we can define a reflection R_α whose action on $\lambda \in E$ is given by $\lambda R_\alpha = \lambda - (\lambda, \alpha^\vee)\alpha$. Suppose Φ is a root system in E. Then the reflections R_α, $\alpha \in \Phi$ generate the finite group called Weyl group $W(\Phi)$ associated with the root system Φ. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the simple roots in Φ then R_{α_i}, $i = 1, 2, \ldots, n$, generate the Weyl group $W(\Phi)$. Let $R_{\alpha_i} = R_i$ for $i = 1, 2, \ldots, n$. Then the elements of the Weyl group $W(\Phi)$ can be written as the product of the generators R_1, $R_2, \ldots, R_n$. In general, for any element σ in W, the expression $\sigma = R_{i_1} R_{i_2} \cdots R_{i_k}$ is not unique. The minimum value of k in all such expressions for a given $\sigma \in W(\Phi)$ is called the length $l(\sigma)$ of σ. There exists a unique element σ_0 in $W(\Phi)$ which has maximum length. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the fundamental weights of Φ. Then we have by definition $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$ (Kronecker delta) for $i, j = 1, 2, \ldots, n$. The action of R_i on λ_j is given by $\lambda_j R_i = \lambda_j - \delta_{ij}\alpha_i$. For $\sigma \in W(\Phi)$, define $I_\sigma = \{\, i \mid 1 \le i \le n,\ l(\sigma R_i) < l(\sigma) \}$. Let $\delta_\sigma = \sum_{i \in I_\sigma} \lambda_i$ and $\varepsilon_\sigma = \delta_\sigma \sigma^{-1}$. We also write W for $W(\Phi)$.

2. The graph on Weyl group

A point $\lambda \in E$ is called W-regular iff λ lies in the interior of a Weyl chamber relative to the root system Φ. It can be shown that the point $\lambda \in E$ is W-regular iff $D(\lambda) \neq 0$ where $D(\lambda)$ is the Weyl's dimension polynomial. This enables us to define a new relation $\longrightarrow$ on W. For σ, $\tau \in W$, define $\sigma \longrightarrow \tau$ iff $-\varepsilon_{\sigma\sigma_0} + \varepsilon_\tau$ is W-regular. The relation $\varepsilon_{\sigma\sigma_0} = -(\delta - \delta_\sigma)\sigma^{-1}$ for $\sigma \in W$, proved in [6], shows that $\sigma \longrightarrow \sigma$. It is shown [6] that only one of $\sigma \longrightarrow \tau$ and $\tau \longrightarrow \sigma$ holds if $\sigma \neq \tau$. We define the graph $\Gamma(W(\Phi))$ whose vertices are the elements of the Weyl group $W(\Phi)$ and for distinct σ, $\tau \in W(\Phi)$ the unordered pair (σ, τ) is an edge iff either $\sigma \longrightarrow \tau$ or $\tau \longrightarrow \sigma$ holds. This gives a graph in the usual sense [9]. The definition of an edge in $\Gamma(W(\Phi))$ shows that the graph depends upon Φ also. We write $\Gamma(W)$ or $\Gamma(\Phi)$ for the graph $\Gamma(W(\Phi))$.

Let J be a subset of $I = \{1, 2, \ldots, n\}$. Then the roots $\{\, \alpha_j \mid j \in J \,\}$ give a root system Φ_J and the group W_J generated by R_j, $j \in J$ is the Weyl group of Φ_J. It is easy to see that $W = W_I$. We have the following result on W_J proved in [3].

Lemma 1. *For distinct σ, $\tau \in W_J$, the unordered pair (σ, τ) is an edge in $\Gamma(W_J)$ iff (σ, τ) is an edge in $\Gamma(W)$. In particular, $\Gamma(W_J)$ is an induced subgraph of $\Gamma(W)$.*

3. Irreducible root systems

In general, a root system Φ is a union of irreducible root systems. So first we consider $\Gamma(\Phi)$ when Φ is an irreducible root system. The irreducible root systems are of the following types: A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4 and G_2.

If the root system is of the type X, we write $\Gamma(X)$ for $\Gamma(\Phi)$. For example $\Gamma(B_2)$ means a graph on a Weyl group whose associated root system is of the type B_2. The fact that the graph $\Gamma(\Phi)$ depends on the root system Φ also and not merely on the Weyl group is best illustrated by the root systems of the type B_n and C_n. The graphs $\Gamma(B_n)$ and $\Gamma(C_n)$ for $n \geq 3$ are distinct although the Weyl groups $W(B_n)$ and $W(C_n)$ are isomorphic.

We describe briefly an interval graph. An intersection graph $\Omega(F)$ for the family $F = \{S_1, S_2, \ldots, S_m\}$ of subsets S_i of a set S is a graph whose vertices are S_1, $S_2, \ldots, S_m$ and for $i \neq j$, S_i is said to be adjacent to S_j iff $S_i \cap S_j$ is not a null set. An interval graph is defined to be a graph which is isomorphic to an intersection graph $\Omega(F)$, where F is some family of intervals on the real line [9]. One can easily replace the real line by any linearly ordered set and the intervals on it in the definition of interval graphs in order to make it more general. In fact, with this definition the characterization of interval graphs have been given by Gilmore and Hoffman [11]. We do not need those characterizations in full form. In fact, it is enough for our purpose to know that a graph G cannot be an interval graph if it has a cycle of length 4 as an induced subgraph [9] The interval graphs have also been studied by Boland and Lekkerkerker [10]. We require some definitions from the graph theory. A cycle in a graph Γ means any finite sequence of vertices $\sigma_1\sigma_2\cdots\sigma_k$ of Γ with the following conditions:

(i) The edges (σ_i, σ_{i+1}) for $1 \leq i \leq k$ are in Γ where $\sigma_{k+1} = \sigma_1$.
(ii) For any two vertices τ and ρ and integers $i, j < k$, $i \neq j$ the relation $\tau = \sigma_i = \sigma_j$, $\rho = \sigma_{i+1} = \sigma_{j+1}$ or $\tau = \sigma_i = \sigma_k$, $\rho = \sigma_{i+1} = \sigma_1$ does not hold.

A cycle $\sigma_1\sigma_2\cdots\sigma_k$ is called odd or even depending on whether k is odd or even. This definition of cycle allows the repetition of vertices i.e. all the vertices in a cycle need not be distinct. Let $\sigma_1\sigma_2\cdots\sigma_k$ be a cycle. Then the edges (σ_i, σ_{i+2}), $1 \leq i \leq k-2$, (σ_{k-1}, σ_1) and (σ_k, σ_2) are called triangular chords of the cycle $\sigma_1\sigma_2\cdots\sigma_k$. With these definitions we have the following theorem of Gilmore and Hoffman: A graph Γ is an interval graph iff every quadrilateral in Γ has a diagonal and every odd cycle in Γ^c, the complementary graph of Γ, has a triangular chord [11].

We show that the graph $\Gamma(\Phi)$ on a Weyl group corresponding to an irreducible root system Φ has a cycle of length 4 as an induced subgraph except in few cases. First we show that a cycle of length 4 occurs as an induced subgraph in the graphs $\Gamma(A_4)$, $\Gamma(B_3)$, $\Gamma(C_3)$, $\Gamma(D_4)$ and $\Gamma(G_2)$. Next we show that for an irreducible root system Φ the graph $\Gamma(\Phi)$, when Φ is not a root system of type A_1, A_2, A_3 and B_2, has one of the graphs $\Gamma(A_4)$, $\Gamma(B_3)$, $\Gamma(C_3)$, $\Gamma(D_4)$ and $\Gamma(G_2)$ as an induced subgraph. Before going into the details of the method we describe briefly the graph $\Gamma(\Phi)$ on a Weyl group for root systems Φ of low orders. The graphs $\Gamma(A_1)$ and $\Gamma(A_2)$ are totally disconnected and have 2 and 6 vertices respectively. The graph $\Gamma(A_3)$ has 24 vertices in which 8 are isolated and has 8 disjoint edges. The graph $\Gamma(B_2)$ has 8 vertices with 4 disjoint edges. This leaves the following graphs on Weyl groups corresponding to an irreducible root system:

$$\left.\begin{array}{lll} \Gamma(A_n) \text{ for } n \geq 4, & \Gamma(B_n) \text{ for } n \geq 3, & \Gamma(C_n) \text{ for } n \geq 3, \\ \Gamma(D_n) \text{ for } n \geq 4, & \Gamma(E_6)\ , & \Gamma(E_7), \\ \Gamma(E_8)\ , & \Gamma(F_4)\ , & \Gamma(G_2). \end{array}\right\} \quad (*)$$

According to the method described before, now we prove the following.

Proposition. *The graphs* $\Gamma(A_4)$, $\Gamma(B_3)$, $\Gamma(C_3)$, $\Gamma(D_4)$ *and* $\Gamma(G_2)$ *are not interval graphs.*

Proof. We show that each of the graphs in the statement has a cycle of length 4 as an induced subgraph. In each case we display the 4 elements of the relevant group which gives a cycle of length 4 as an induced subgraph. We give a table displaying the elements σ of a Weyl group along with ε_σ and $\varepsilon_{\sigma\sigma_0}$. It is easy to verify that $D(-\varepsilon_{\sigma\sigma_0} + \varepsilon_\tau)$ is zero or not for a given pair of elements σ, τ in this table. The elements of the Weyl group are of the form $R_{i_1}R_{i_2}\cdots R_{i_k}$ and for convenience we write this as $i_1i_2\cdots i_k$. For example we write 2312 for $R_2R_3R_1R_2$. The identity element of a Weyl group is written as *id*.

(i) The graph $\Gamma(A_4)$ has 120 vertices and 180 edges. For $\lambda \in E$ we have $\lambda = x\lambda_1 + y\lambda_2 + z\lambda_3 + t\lambda_4$ and the Weyl's dimension polynomial $D(\lambda)$ in this case is $\frac{\phi(x,y,z,t)}{\phi(1,1,1,1)}$ where

$$\phi(x,y,z,t) = xyzt(x+y)(y+z)(z+t)(x+y+z)(y+z+t)(x+y+z+t).$$

The four elements σ_1, σ_2, σ_3 and σ_4 which give the edges (σ_1, σ_2), (σ_2, σ_3), (σ_3, σ_4) and (σ_4, σ_1) making a cycle of length 4 as an induced subgraph are listed below. We conclude that $\Gamma(A_4)$ is not an interval graph.

S. No.	σ	ε_σ	$\varepsilon_{\sigma\sigma_0}$
1	$\sigma_1 = id$	0	$-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$
2	$\sigma_2 = 324$	$\lambda_1 + \lambda_2 - 2\lambda_3 + \lambda_4$	$-\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4$
3	$\sigma_3 = 4321$	$-\lambda_4$	$-\lambda_1 - \lambda_2 - \lambda_3 + 3\lambda_4$
4	$\sigma_4 = 14232$	$-2\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_4$	$\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4$

(ii) The graph $\Gamma(B_3)$ has 48 vertices and 100 edges. For $\lambda \in E$, $\lambda = x\lambda_1 + y\lambda_2 + z\lambda_3$ and the Weyl's dimension polynomial $D(\lambda)$ is given by $\frac{\phi(x,y,z)}{\phi(1,1,1)}$ where

$$\phi(x,y,z) = xyz(x+y)(y+z)(2y+z)(x+y+z)(x+2y+z)(2x+2y+z).$$

The four elements σ_1, σ_2, σ_3 and σ_4 which give the edges (σ_1, σ_2), (σ_2, σ_3), (σ_3, σ_4) and (σ_4, σ_1) making a cycle of length 4 are listed below. This cycle is also an induced subgraph. It shows that $\Gamma(B_3)$ is not an interval graph.

S. No.	σ	ε_σ	$\varepsilon_{\sigma\sigma_0}$
1	$\sigma_1 = id$	0	$-\lambda_1 - \lambda_2 - \lambda_3$
2	$\sigma_2 = 321$	$\lambda_2 - 2\lambda_3$	$-\lambda_1 - 2\lambda_2 + 3\lambda_3$
3	$\sigma_3 = 2323$	$3\lambda_1 - \lambda_2 - \lambda_3$	$-\lambda_1$
4	$\sigma_4 = 213$	$\lambda_1 - 2\lambda_2 + 3\lambda_3$	$-\lambda_1 + \lambda_2 - 2\lambda_3$

(iii) The graph $\Gamma(C_3)$ has 48 vertices and 96 edges. For $\lambda \in E$, $\lambda = x\lambda_1 + y\lambda_2 + z\lambda_3$ and the Weyl's dimension polynomial $D(\lambda)$ is given by $\frac{\phi(x,y,z)}{\phi(1,1,1)}$ where

$$\phi = xyz(x+y)(y+z)(y+2z)(x+y+z)(x+y+2z)(x+2y+2z).$$

We give below the 4 elements σ_1, σ_2, σ_3 and σ_4 which make a cycle of length 4 whose edges are (σ_1, σ_2), (σ_2, σ_3), (σ_3, σ_4) and (σ_4, σ_1) and which is also an induced subgraph of $\Gamma(C_3)$. This proves that $\Gamma(C_3)$ is not an interval graph.

S. No.	σ	ε_σ	$\varepsilon_{\sigma\sigma_0}$
1	$\sigma_1 = id$	0	$-\lambda_1 - \lambda_2 - \lambda_3$
2	$\sigma_2 = 2132$	$\lambda_1 - 2\lambda_2 + \lambda_3$	$-2\lambda_1 + 3\lambda_2 - 2\lambda_3$
3	$\sigma_3 = 1231213$	$-\lambda_1 - 2\lambda_2 + \lambda_3$	$\lambda_1 + \lambda_2 - \lambda_3$
4	$\sigma_4 = 1232$	$-2\lambda_1 - \lambda_2$	$3\lambda_1 - \lambda_2 - \lambda_3$

(iv) The graph $\Gamma(D_4)$ has 192 vertices and 624 edges. For $\lambda \in E$, $\lambda = x\lambda_1 + y\lambda_2 + z\lambda_3 + t\lambda_4$ and the Weyl's dimension polynomial $D(\lambda)$ is given by $\frac{\phi(x,y,z,t)}{\phi(1,1,1,1)}$ where

$$\phi(x,y,z,t) = xyzt(x+y)(y+z)(y+t)(x+y+z)(x+y+t)(y+z+t)(x+y+z+t)(x+2y+z+t).$$

The required 4 elements σ_1, σ_2, σ_3 and σ_4 are listed below. These give an induced subgraph of $\Gamma(D_4)$ which is a cycle of length 4 with edges (σ_1, σ_2), (σ_2, σ_3), (σ_3, σ_4) and (σ_4, σ_1). This shows that $\Gamma(D_4)$ is not an interval graph.

S. No.	σ	ε_σ	$\varepsilon_{\sigma\sigma_0}$
1	$\sigma_1 = id$	0	$-\lambda_1-\lambda_2-\lambda_3-\lambda_4$
2	$\sigma_2 = 24312$	$\lambda_1-2\lambda_2+\lambda_3+\lambda_4$	$-2\lambda_1+3\lambda_2-2\lambda_3-2\lambda_4$
3	$\sigma_3 = 324312134$	$\lambda_1-2\lambda_2-\lambda_3+\lambda_4$	$-\lambda_1+\lambda_2+\lambda_3-\lambda_4$
4	$\sigma_4 = 32412$	$\lambda_2-2\lambda_3$	$-\lambda_1-\lambda_2+3\lambda_3-\lambda_4$

(v) The graph $\Gamma(G_2)$ has 12 vertices and 12 edges. For $\lambda \in E$, $\lambda = x\lambda_1 + y\lambda_2$ and the Weyl's dimension polynomial $D(\lambda)$ is given by $\frac{\phi(x,y)}{\phi(1,1)}$ where

$$\phi = xy(x+y)(x+2y)(x+3y)(2x+3y).$$

We list below 4 elements σ_1, σ_2, σ_3 and σ_4. These elements give a cycle of length 4 with edges (σ_1, σ_2), (σ_2, σ_3), (σ_3, σ_4) and (σ_4, σ_1) which is also an induced subgraph of $\Gamma(G_2)$.

S. No.	σ	ε_σ	$\varepsilon_{\sigma\sigma_0}$
1	$\sigma_1 = id$	0	$-\lambda_1-\lambda_2$
2	$\sigma_2 = 121$	$-2\lambda_1+\lambda_2$	$3\lambda_1-2\lambda_2$
3	$\sigma_3 = 121212$	$-\lambda_1-\lambda_2$	0
4	$\sigma_4 = 212$	$3\lambda_1-2\lambda_2$	$-2\lambda_1+\lambda_2$

Therefore we conclude that $\Gamma(G_2)$ is not an interval graph. ☑

We now come to our main result.

Theorem 1. *Let Φ be an irreducible root system. The only graphs $\Gamma(\Phi)$ on the Weyl groups which are interval graphs are $\Gamma(A_1)$, $\Gamma(A_2)$, $\Gamma(A_3)$ and $\Gamma(B_2)$.*

Proof. Recall that the graphs $\Gamma(A_1)$, $\Gamma(A_2)$, $\Gamma(A_3)$ and $\Gamma(B_2)$ have the connected components as isolated vertices and disjoint edges. An isolated vertex and a disjoint edge are the interval graphs of a single interval and two intersecting intervals on a real line respectively. Therefore, the union of appropriate number of disjoint single intrevals and two intersecting intervals on a real line gives the graphs on $\Gamma(A_1)$, $\Gamma(A_2)$, $\Gamma(A_3)$ and $\Gamma(B_2)$. This proves that these graphs are interval graphs.

Next we show that the remaining graphs on Weyl groups, which are precisely those listed in $(*)$, are not interval graphs. Our method of proof is to show that each of the graphs in $(*)$ has one of the graphs $\Gamma(A_4)$, $\Gamma(B_3)$, $\Gamma(C_3)$, $\Gamma(D_4)$ and $\Gamma(G_2)$ as an induced subgraph. Then by Lemma 1, and the proposition it follows that the graphs in $(*)$ are not interval graphs. For each of the graph in $(*)$ we exhibit a proper choice of J so that $\Gamma(W_J)$ is one of the graphs in the proposition. For the explicit choice of J we refer to the Dynkin diagrams of the irreducible root system given in [9, p. 58]. For the graph $\Gamma(A_n)$ for $n \geq 4$ choose $J = \{1,2,3,4\}$ to get $\Gamma(A_4)$ as an induced subgraph. In the case of $\Gamma(B_n)$ for $n \geq 3$ and $\Gamma(F_4)$ choose $J = \{n-2, n-1, n\}$ and $J = \{1,2,3\}$

respectively to get an induced subgraph $\Gamma(B_3)$. The graph $\Gamma(C_3)$ is an induced subgraph of $\Gamma(C_n)$ for $n \geq 3$ with $J = \{n-2, n-1, n\}$. The graphs $\Gamma(D_n)$ for $n \geq 4$, $\Gamma(E_6)$, $\Gamma(E_7)$ and $\Gamma(E_8)$ have $\Gamma(D_4)$ as an induced subgraph with $J = \{n-3, n-2, n-1, n\}$ for $\Gamma(D_n)$ and $J = \{2, 3, 4, 5\}$ for the rest. The graph $\Gamma(G_2)$ is not an interval graph by the proposition. This complete the proof. ☑

4. Arbitrary root systems

Let Φ be a root system. If Φ is not irreducible then it is a union of irreducible root systems. It is known that the Dynkin diagram of Φ is connected iff Φ is irreducible root system. If Φ is reducible and is union of irreducible root systems Φ_1, Φ_2, ... , Φ_k then the connected components of the Dynkin diagram of Φ are precisely the Dynkin diagram of each irreducible component Φ_i of Φ. The Dynkin diagram also determines the Weyl group uniquely. The Weyl group $W(\Phi)$ is the direct product of the Weyl groups $W(\Phi_i)$, $i = 1, 2, \ldots, k$. If any one of the root system Φ_i is of the type given in $(*)$ then the graph $\Gamma(\Phi)$ cannot be an interval graph by Lemma 1. Therefore we can assume that Φ is a union of the root systems of the type A_1, A_2, A_3 and B_2 where repetitions are allowed. We require some results to analyze such a root system Φ. Suppose Φ is a union of two root systems Φ_1 and Φ_2 which are not necessarily irreducible. In this case we write $\Phi = \Phi_1 \times \Phi_2$ and if $\Phi_1 = \Phi_2$ then $\Phi = (\Phi_1)^2$. The Weyl group W of Φ is the direct product of the Weyl groups W_1 and W_2 of the root systems Φ_1 and Φ_2 respectively. Since $W = W_1 \times W_2$ (direct product), every element $\rho \in W$ can be written uniquely as $\rho = \sigma\tau$ with $\sigma \in W_1$ and $\tau \in W_2$. Also $I_\rho = I_\sigma \cup I_\tau$ (disjoint union) and $\varepsilon_\rho = \varepsilon_\sigma \oplus \varepsilon_\tau$ (direct sum). This shows that $\varepsilon_{\sigma\tau} = \varepsilon_\sigma \oplus \varepsilon_\tau$ for $\sigma \in W_1$ and $\tau \in W_2$. If δ, δ_1 and δ_2 are the fundamental weights of the root systems Φ, Φ_1 and Φ_2 respectively then $\delta = \delta_1 \oplus \delta_2$. It is easy to see that if σ_0, σ_0' and σ_0'' are the unique elements of maximal length in W, W_1 and W_2 respectively then $\sigma_0 = \sigma_0'\sigma_0''$. Similar relations hold if Φ is a union of more than two root systems.

With above notations we prove the following results.

Lemma 2. *Let σ_1, $\sigma_2 \in W_1$ and τ_1, $\tau_2 \in W_2$. The relations $\sigma_1 \longrightarrow \sigma_2$ in W_1 and $\tau_1 \longrightarrow \tau_2$ in W_2 hold iff $\sigma_1\tau_1 \longrightarrow \sigma_2\tau_2$ in $W = W_1 \times W_2$ holds. If $\sigma_1 \longrightarrow \sigma_2$ in W_1 with $\sigma_1 \neq \sigma_2$ and $\tau_1 \longrightarrow \tau_2$ in W_2 with $\tau_1 \neq \tau_2$ then $\sigma_1\tau_2 \not\twoheadrightarrow \sigma_2\tau_1$ and $\sigma_2\tau_1 \not\twoheadrightarrow \sigma_1\tau_2$ in W.*

Proof. For the first part of the statement see [5]. The proof for the second is by contradiction. Suppose $\sigma_1\tau_2 \longrightarrow \sigma_2\tau_1$. The result in the first part implies that $\sigma_1 \longrightarrow \sigma_2$ in W_1 and $\tau_2 \longrightarrow \tau_1$ in W_2. But by assumption $\tau_1 \longrightarrow \tau_2$ in

W_2. But both $\tau_1 \longrightarrow \tau_2$ and $\tau_2 \longrightarrow \tau_1$ in W_2 cannot be true [6]. Therefore $\sigma_1\tau_2 \nrightarrow \sigma_2\tau_1$. Similarely $\sigma_2\tau_1 \nrightarrow \sigma_1\tau_2$ can be proved. ☑

Remark. The lemma can be generalized when $W = W_1 \times W_2 \times \cdots \times W_k$.

Lemma 3. *Let (σ_1, σ_2) be an edge in $\Gamma(W_1)$ and (τ_1, τ_2) be an edge in $\Gamma(W_2)$. Then the induced subgraph for the 4 vertices $\sigma_i\tau_j$, $i, j = 1, 2$, in $\Gamma(W_1 \times W_2)$ is a quadrilateral with a diagonal. More precisely, if $\sigma_1 \longrightarrow \sigma_2$ in W_1 and $\tau_1 \longrightarrow \tau_2$ in W_2 then following are the only edges joining the vertices $\sigma_i\tau_j$, $i, j = 1, 2$ in $\Gamma(W_1 \times W_2)$: $(\sigma_1\tau_1, \sigma_2\tau_1)$, $(\sigma_1\tau_2, \sigma_2\tau_2)$, $(\sigma_1\tau_1, \sigma_1\tau_2)$, $(\sigma_2\tau_1, \sigma_2\tau_2)$ and $(\sigma_1\tau_1, \sigma_2\tau_2)$.*

Proof. If W is any Weyl group then for $\sigma \in W$ we always have $\sigma \longrightarrow \sigma$ [6]. Therefore, $\sigma_i \longrightarrow \sigma_i$ for $i = 1, 2$ in W_1 and $\tau_i \longrightarrow \tau_i$ for $i = 1, 2$ in W_2. The result follows by applying Lemma 2 to $\sigma_1 \longrightarrow \sigma_2$ in W_1, $\tau_1 \longrightarrow \tau_2$ in W_2 and to $\sigma_i \longrightarrow \sigma_i$ for $i = 1, 2$ in W_1 and $\tau_i \longrightarrow \tau_i$ for $i = 1, 2$ in W_2. ☑

In a graph Γ, we shall call a vertex isolated if it is not adjacent to any other vertex and an edge disjoint if it is not adjacent to any other edge. With these definitions we prove the following.

Lemma 4. *Let $\sigma \in W_1$ and $\tau \in W_2$. Then σ and τ are isolated vertices in $\Gamma(W_1)$ and $\Gamma(W_2)$ respectively iff $\sigma\tau$ is an isolated vertex in $\Gamma(W_1 \times W_2)$.*

Proof. We show that if neither σ is an isolated vertex in $\Gamma(W_1)$ nor τ is an isolated vertex in $\Gamma(W_2)$ then $\sigma\tau$ is not an isolated vertex in $\Gamma(W_1 \times W_2)$. Suppose σ is not an isolated vertex in $\Gamma(W_1)$. Then for some $\sigma_1 \in W_1$, (σ, σ_1) is an edge in $\Gamma(W_1)$. By Lemma 2, $(\sigma\tau, \sigma_1\tau)$ is an edge in $\Gamma(W_1 \times W_2)$ as $\sigma \longrightarrow \sigma_1$ in W_1. This shows that $\sigma\tau$ is not an isolated vertex. Same result can be proved if we assume that τ is not an isolated vertex in $\Gamma(W_2)$.

Conversely, suppose $\sigma\tau$ is not an isolated vertex in $\Gamma(W_1 \times W_2)$. Then for some $\rho \in W_1 \times W_2$, $\rho \neq \sigma\tau$, the unordered pair $(\sigma\tau, \rho)$ is an edge in $\Gamma(W_1 \times W_2)$. Now $\rho \in W_1 \times W_2$ implies that $\rho = \sigma_1\tau_1$ for unique $\sigma_1 \in W_1$ and $\tau_1 \in W_2$. Therefore, $(\sigma\tau, \sigma_1\tau_1)$ is an edge in $\Gamma(W_1 \times W_2)$ and this gives $\sigma\tau \longrightarrow \sigma_1\tau_1$ which in turn gives $\sigma \longrightarrow \sigma_1$ in W_1 and $\tau \longrightarrow \tau_1$ in W_2. Since $\rho = \sigma_1\tau_1 \neq \sigma\tau$, we must have either $\sigma \neq \sigma_1$ or $\tau \neq \tau_1$. Therefore, either $\sigma \longrightarrow \sigma_1$ with $\sigma \neq \sigma_1$ or $\tau \longrightarrow \tau_1$ with $\tau \neq \tau_1$. We conclude that either (σ, σ_1) is an edge in $\Gamma(W_1)$ or (τ, τ_1) is an edge in $\Gamma(W_2)$. This shows that either σ is not an isolated vertex in $\Gamma(W_1)$ or τ is not an isolated vertex in $\Gamma(W_2)$. ☑

Corollary. *The graphs $\Gamma(W_1)$ and $\Gamma(W_2)$ are totally disconnected if and only if $\Gamma(W_1 \times W_2)$ is totally disconnected.*

Remark. This corollary has obvious generalization to the direct product of more than two Weyl groups.

Next we need a result proved in [3] which is stated below.

Lemma 5. *If $\Gamma(W_1)$ is totally disconnected and $\Gamma(W_2)$ is any graph on a Weyl group W_2 then $\Gamma(W_1 \times W_2)$ consists of $|W_1|$ number of disjoint copies of the graph $\Gamma(W_2)$.*

The following result will be required in the proof of our main result.

Lemma 6. *Suppose the graphs $\Gamma(W_1)$ and $\Gamma(W_2)$ are such that their connected components are isolated vertices or disjoint edges or both. Then the graph $\Gamma(W_1 \times W_2)$ is an interval graph.*

Proof. We have shown in Lemma 4, that if $\sigma \in W_1$ and $\tau \in W_2$ are isolated vertices in $\Gamma(W_1)$ and $\Gamma(W_2)$ then $\sigma\tau$ is an isolated vertex in $\Gamma(W_1 \times W_2)$.

Next we prove that an isolated vertex $\sigma \in W_1$ in $\Gamma(W_1)$ and a disjoint edge (τ_1, τ_2) in $\Gamma(W_2)$ for $\tau_1, \tau_2 \in W_2$ gives a disjoint edge $(\sigma\tau_1, \sigma\tau_2)$ in $\Gamma(W_1 \times W_2)$. Suppose $(\sigma\tau_1, \sigma\tau_2)$ is not a disjoint edge in $\Gamma(W_1 \times W_2)$. Then either for some $\rho \neq \sigma\tau_2$, the pair $(\sigma\tau_1, \rho)$ or for $\rho \neq \sigma\tau_1$ the pair $(\sigma\tau_2, \rho)$ is an edge in $\Gamma(W_1 \times W_2)$. Suppose $(\sigma\tau_1, \rho)$ with $\rho \neq \sigma\tau_2$ is an edge in $\Gamma(W_1 \times W_2)$. This gives $\rho \in W_1 \times W_2$ and $\rho = \sigma_1\tau_3$ with unique $\sigma_1 \in W_1$ and $\tau_3 \in W_2$ and $\rho \neq \sigma\tau_1$ and $\rho \neq \sigma\tau_2$ i.e. $\sigma\tau_1 \neq \sigma_1\tau_3$ and $\sigma\tau_2 \neq \sigma_1\tau_3$. The last part shows that either $\sigma \neq \sigma_1$ or $\tau_1 \neq \tau_3$ and $\tau_2 \neq \tau_3$. If $(\sigma\tau_1, \sigma_1\tau_3)$ is an edge in $\Gamma(W_1 \times W_2)$ then $\sigma\tau_1 \longrightarrow \sigma_1\tau_3$. By Lemma 2, $\sigma \longrightarrow \sigma_1$ and $\tau_1 \longrightarrow \tau_3$. Therefore, if $\sigma \neq \sigma_1$ then (σ, σ_1) is an edge in $\Gamma(W_1)$ and if $\tau_1 \neq \tau_3$ then (τ_1, τ_3) where $\tau_3 \neq \tau_2$ is an edge in $\Gamma(W_2)$. This shows that either σ is not an isolated vertex or (τ_1, τ_2) is not a disjoint edge. A contradiction. Similar contradiction is arrived at if $(\sigma\tau_2, \rho)$ with $\rho \neq \sigma\tau_1$ is an edge in $\Gamma(W_1 \times W_2)$. We conclude that $(\sigma\tau_1, \sigma\tau_2)$ is a disjoint edge in $\Gamma(W_1 \times W_2)$.

Next suppose (σ_1, σ_2) and (τ_1, τ_2) are disjoint edges in $\Gamma(W_1)$ and $\Gamma(W_2)$ respectively. Lemma 3 shows that the 4 vertices $\sigma_i\tau_j$, $i, j = 1, 2$ in $\Gamma(W_1 \times W_2)$ give a quadrilateral with a diagonal. Again we can show that this quadrilateral with a diagonal is a connected component of the graph $\Gamma(W_1 \times W_2)$ by the arguments similar to those used in the previous paragraph.

This shows that a connected component of the graph $\Gamma(W_1 \times W_2)$ is either a vertex or an edge or else a quadrilateral with a diagonal. The first two are interval graphs where we can take a single interval or two intersecting intervals on the real line. For the last one take intervals as given below: let $x_1, x_2, \ldots, x_8$ be real number with $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8$ and $a = [x_3, x_7]$, $b = [x_5, x_8]$, $c = [x_2, x_6]$ and $d = [x_1, x_4]$ be the 4 intervals. These give a quadrilateral *abcd* with diagonal *ac*. This proves that $\Gamma(W_1 \times W_2)$ is an interval graph. ☑

Lemma 7. *Let Φ_1, Φ_2 and Φ_3 be root systems such that each of $\Gamma(\Phi_1)$, $\Gamma(\Phi_2)$ and $\Gamma(\Phi_3)$ has at least one edge. Then $\Gamma(\Phi_1 \times \Phi_2 \times \Phi_3)$ is not an interval graph.*

Proof. Let $W_i = W(\Phi_i)$, $i = 1, 2, 3$. Suppose σ_1, $\sigma_2 \in W_1$; τ_1, $\tau_2 \in W_2$; ρ_1, $\rho_2 \in W_3$ and (σ_1, σ_2), (τ_1, τ_2) and (ρ_1, ρ_2) are edges in $\Gamma(\Phi_1)$, $\Gamma(\Phi_2)$ and $\Gamma(\Phi_3)$ respectively. Therefore, we also have $\sigma_1 \longrightarrow \sigma_2$ in W_1, $\tau_1 \longrightarrow \tau_2$ in W_2 and $\rho_1 \longrightarrow \rho_2$ in W_3. If $W = W(\Phi_1 \times \Phi_2 \times \Phi_3)$ then the 8 elements $\sigma_i\tau_j\rho_k$, $i, j, k = 1, 2$ are in W. Put $\sigma_1\tau_1\rho_1 = o$, $\sigma_2\tau_1\rho_1 = a$, $\sigma_1\tau_2\rho_1 = b$, $\sigma_1\tau_1\rho_2 = c$, $\sigma_2\tau_2\rho_1 = d$, $\sigma_1\tau_2\rho_2 = e$, $\sigma_2\tau_1\rho_2 = f$, $\sigma_2\tau_2\rho_2 = g$. By the repeated application of the Lemma 2 to the edges (σ_1, σ_2), (τ_1, τ_2) and (ρ_1, ρ_2) we get the following 19 edges in $\Gamma(W)$ i.e., $\Gamma(\Phi_1 \times \Phi_2 \times \Phi_3)$: oa, ob, oc, od, oe, of, og, ce, be, cf, af, cg, dg, bd, eg, bg, ad, fg and ag. The induced subgraph Γ_1 of these 8 vertices o, a, b, c, d, e, f and g in $\Gamma(W)$ is shown in Figure 1.

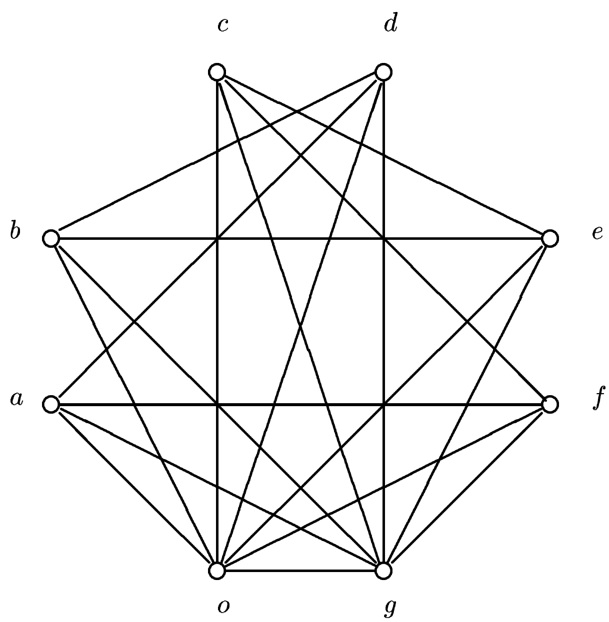

FIGURE 1. *The graph* Γ_1.

It is easy to see that every quadrilateral in Γ_1 has a diagonal. In any case, we consider the graph Γ_1^c, the complementary graph of Γ_1, shown in Figure 2.

The graph Γ_1^c has an odd cycle $\alpha = eabfbcdca$ which has no triangular chord. The cycle α is also an odd cycle with no triangular chord in $\Gamma^c(W)$ as Γ_1 is an induced subgraph of $\Gamma(W)$. Therefore, by the theorem of Gilmore and Hoffman [11] the graph $\Gamma(W)$ i.e. $\Gamma(\Phi_1 \times \Phi_2 \times \Phi_3)$ is not an interval graph. ☑

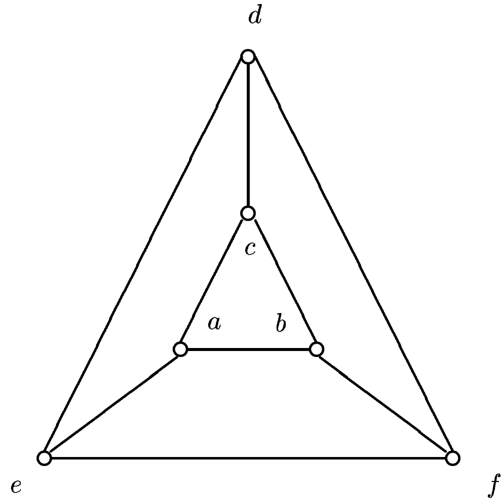

FIGURE 2. *The graph Γ_1^c.*

Theorem 2. *Let Φ be any root system. The graph $\Gamma(\Phi)$ is an interval graph iff Φ is equal to Φ_1 or Φ_2 or else $\Phi_1 \times \Phi_2$ where Φ_1 is a root system of type $(A_1)^{k_1} \times (A_2)^{k_2}$ where k_1, k_2 are nonnegative integers with at least one of k_1, k_2 nonzero, and Φ_2 is any one of the following root systems: A_3, B_2, $A_3 \times A_3$, $A_3 \times B_2$ and $B_2 \times B_2$.*

Proof. The graphs $\Gamma(A_1)$ and $\Gamma(A_2)$ are totally disconnected. By Lemma 4, the graph $\Gamma(A_1^{k_1} \times A_2^{k_2})$ is also totally disconnected. Therefore $\Gamma(\Phi_1)$ is an interval graph as an isolated vertex can be obtained by a single interval on a real line. As mentioned earlier we have shown that $\Gamma(A_3)$ and $\Gamma(B_2)$ are interval graphs. The connected components of the graphs $\Gamma(A_3)$ and $\Gamma(B_2)$ are isolated vertices and disjoint edges. Therefore by Lemma 6, the graphs $\Gamma(A_3 \times A_3)$, $\Gamma(A_3 \times B_2)$ and $\Gamma(B_2 \times B_2)$ are interval graphs. This shows that $\Gamma(\Phi_2)$ is an interval graph. Next, the graph $\Gamma(\Phi_1)$ is totally disconnected and hence by Lemma 5 the graph $\Gamma(\Phi_1 \times \Phi_2)$ has $|W(\Phi_1)|$ number of disjoint copies of the graph $\Gamma(\Phi_2)$. Here $|W(\Phi_1)|$ is the number of elements in $W(\Phi_1)$ i.e. number of vertices in $\Gamma(\Phi_1)$. This shows that the graph $\Gamma(\Phi_1 \times \Phi_2)$ is an interval graph as $\Gamma(\Phi_2)$ is an interval graph.

Next we prove the converse. The Theorem 1 shows that if Φ contains any root system given in $(*)$, then $\Gamma(\Phi)$ is not an interval graph by Lemma 1. Suppose Φ does not contain any of the root systems given in $(*)$ and Φ_2. Then Φ is union of at least 3 root systems of type A_3 and B_2 with repetitions. Since $\Gamma(A_3)$ and $\Gamma(B_2)$ are not totally disconnected, by Lemma 7 the graph $\Gamma(\Phi)$ is not an interval graph. ☑

References

[1] Samy A. Youssef, *The graphs on Weyl groups,* Ph.D. Thesis, Indian Institute of Technology, Kharagpur, 1992.

[2] Samy A. Youssef & S. G. Hulsurkar, *On automorphisms of a graph on Weyl groups,* J. Math. Phy. Sci. **26** (1992), 419–423.

[3] Samy A. Youssef & S. G. Hulsurkar, *On planarity of graphs on Weyl groups,* Tamkang Journal of Mathematics **26**(1995), 365–373.

[4] Samy A. Youssef & S. G. Hulsurkar, *On connectedness of graphs on Weyl groups of type* $A_n (n \geq 4)$*,* Archivum Mathematicum **31** (1995), 163–170.

[5] Samy A. Youssef & S. G. Hulsurkar, *More on the girth of the graphs on Weyl groups,* Archivum Mathematicum **29** (1993), 19–23.

[6] S. G. Hulsurkar, *Proof of Verma's conjecture on Weyl's dimension polynomial,* Inventiones Math. **27** (1974), 45–52.

[7] L. Chaskofsky, *Variation on Hulsurkar's matrix with applications to representation of algebraic Chevally groups,* J. Algebra **82** (1983), 255–274.

[8] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory,* Springer Verlag, New York, 1972.

[9] F. Harary, *Graph Theory*, Addison Wesley, 1972.

[10] C. G. Lekkerkerker & J. Ch. Boland, *Representation of a finite graph by a set of intervals in the real line,* Fund. Math. **51** (1962), 45–64.

[11] P. C. Gilmore & A. J. Hoffman, *A characterrization of comparability graphs and of interval graphs,* Canad. J. Math. **16** (1964), 539–548.

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