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Nontrivial solitary waves of GKP equation in multi-dimensional spaces

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Abstract. In this paper, using the Mountain Pass Lemma without (PS) condition due to Ambrosetti and Rabinowitz, we obtain the existence of the nontrivial solitary waves of Generalized Kadomtsev-Petviashvili equation in multi-dimensional spaces and for superlinear nonlinear term $f(u)$ which satisfies some growth condition. By the Pohozaev type variational identity, we obtain the nonexistence of the nontrivial solitary waves for power function nonlinear case, i.e. $f(u) = u^p$ where $p \geq 2(2n-1)/(2n-3)$.

Keywords and phrases. Mountain Pass Lemma, Solitary wave, Generalized Kadomtsev-Petviashvili equation.

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1. Introduction

In this paper, we shall investigate the existence and nonexistence of the nontrivial solitary waves of Generalized Kadomtsev-Petviashvili equation in multi-dimensional spaces

$$w_t + w_{xxx} + (f(w))_x = D_x^{-1}\Delta_y w, \tag{1.1}$$

where $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{n-1}$, $n \geq 2$, $D_x^{-1}h(x, y) = \int_{-\infty}^{x} h(s, y)ds$ and $\Delta_y := \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2}$.

Kadomtsev-Petviashvili equation and its generalization appear in many Physic progress (cf. [3], [4], [5], [6], [7] and the references therein). A solitary wave

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is a solution of the form

$$w(t,x,y) = u(x-ct,y),$$

where $c>0$ is fixed. Substituting in (1.1), there holds

$$-cu_x + u_{xxx} + (f(u))_x = D_x^{-1}\Delta_y u,$$

or

$$\big(-u_{xx} + D_x^{-2}\Delta_y u + cu - f(u)\big)_x = 0. \tag{1.2}$$

In [4] and [5], using constrained minimization, De Bouard and Saut obtained the existence and nonexistence of solitary waves in the case where power nonlinearities $f(u)=u^p,\ p=m/n,\ m,n$ are relatively prime, n is odd. In Chapter 7 of [7], Willem extended the results of [4] to the case where $n=2,\ f(u)$ is a continuous function satisfying some structure conditions.

In this paper we mainly deal with the case where $n\ge 2$ and $f(u)$ is a continuous function. The rest of this paper is organized as: §2 gives the functional setting of the problem and some embedding theorems which will be used latter; §3 deals with the existence of the nontrivial solitary waves. In §4, first we derive a variational identity and then use this identity to prove the nonexistence of the nontrivial solitary waves.

2. **Preliminaries**

In order to attack the existence and nonexistence of the nontrivial solitary waves of problem (1.1) we apply the following functional setting:

Definition 2.1. On $Y := \{g_x \mid g\in \mathcal{D}(\mathbb{R}^n)\}$, we define the inner product

$$(u,v) := \int_{R^n} \big[u_x v_x + D_x^{-1}\nabla_y u\cdot D_x^{-1}\nabla_y v + cuv\big]\,dV, \tag{2.1}$$

where $\nabla_y = (\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_{n-1}})$, $dV = dxdy$, and the corresponding norm

$$\|u\| := \Big(\int_{R^n}\big[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2\big]\,dV\Big)^{1/2}. \tag{2.2}$$

A function $u:\mathbb{R}^n\to\mathbb{R}$ belongs to X if there exists $\{u_m\}_{m=1}^{+\infty}\subset Y$ such that:

(a) $u_m\to u$ a.e. on $\mathbb{R}^n$;
(b) $\|u_j-u_k\|\to 0$ as $j,k\to\infty$.

Note that the space X with inner product (2.1) and norm (2.2) is a Hilbert space.

We will show that if estimate

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\Big(\int_{\mathbb{R}^n}\big[u_x^2 + |D_x^{-1}\nabla_y u|^2\big]\,dV\Big)^{1/2} \tag{2.3}$$

holds for a certain constant $C > 0$ and all functions $u \in Y$, there is only one possibility: $q = \bar{p} = \frac{2(2n-1)}{2n-3}$. In fact, let $u \in Y, u \not\equiv 0$, and define for $\lambda > 0$ the rescaled function

$$u_\lambda(x,y) = u(\lambda x, \lambda^2 y),\ (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Applying (2.3) to u_λ, there holds

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C\Big(\int_{\mathbb{R}^n} \big[(u_\lambda)_x^2 + |D_x^{-1}\nabla_y u_\lambda|^2\big]\, dV\Big)^{1/2}. \tag{2.4}$$

But simple computation implies

$$\int_{\mathbb{R}^n} |u_\lambda|^q \, dV = \frac{1}{\lambda^{2n-1}} \int_{\mathbb{R}^n} |u|^q \, dV, \tag{2.5}$$

$$\int_{\mathbb{R}^n} (u_\lambda)_x^2 \, dV = \frac{1}{\lambda^{2n-3}} \int_{\mathbb{R}^n} u_x^2 \, dV, \tag{2.6}$$

and

$$\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u_\lambda|^2 \, dV = \frac{1}{\lambda^{2n-3}} \int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2 \, dV. \tag{2.7}$$

Inserting these equalities into (2.4), there holds

$$\frac{1}{\lambda^{(2n-1)/q}}\|u\|_{L^q(\mathbb{R}^n)} \le C\frac{1}{\lambda^{(2n-3)/2}}\Big(\int_{\mathbb{R}^n} \big[u_x^2 + |D_x^{-1}\nabla_y u|^2\big]\, dV\Big)^{1/2}.$$

That is

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\lambda^{\frac{2n-1}{q}-\frac{2n-3}{2}}\Big(\int_{\mathbb{R}^n} \big[u_x^2 + |D_x^{-1}\nabla_y u|^2\big]\, dV\Big)^{1/2} \tag{2.8}$$

But then if $\frac{2n-1}{q} - \frac{2n-3}{2} \neq 0$, upon sending λ to either 0 or ∞ in (2.8), we can obtain a contradiction. Thus the only possibility is that $\frac{2n-1}{q} - \frac{2n-3}{2} = 0$, i.e, $q = \bar{p} = \frac{2(2n-1)}{2n-3}$.

Actually, from the embedding theorems for anisotropic Sobolev spaces(cf. [2], p. 323), the following lemma asserts that (2.3) holds if and only if $q = \bar{p}$.

Lemma 2.2. *If $q = \bar{p} = \frac{2(2n-1)}{2n-3}$, there exists a constant $C > 0$ such that (2.3) holds for all functions $u \in X$.*

From the interpolation theorem and estimate (2.3), there is an embedding theorem about X as follows:

Lemma 2.3. *The following embeddings are continuous:*

$$X \hookrightarrow L^p(\mathbb{R}^n), 2 \le p \le \bar{p}.$$

Lemma 2.4. *The following embeddings are compact:*

$$X \hookrightarrow\hookrightarrow L^p_{loc}(\mathbb{R}^n), 2 \le p < \bar{p}.$$

Proof. Suppose that $\{u_m\}_{m=1}^{\infty} \subset X$ is bounded in norm (2.2). Without loss of generality, assume that there exists $\{g_m\}_{m=1}^{\infty} \subset L^2_{loc}(R^n)$ such that $u_m = \partial_x g_m$. Let $v_m = (v_{m,1},\ v_{m,2},\ \cdots,\ v_{m,n-1}) = \nabla_y g_m \in (L^2(R^n))^{n-1}$.

Multiplying g_m by $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \le \psi \le 1,\ \psi \equiv 1$ on $B(0,R)$ and $\operatorname{supp}\psi \subset B(0,2R)$, we may assume that $\operatorname{supp} g_m \subset B(0,2R)$. Selecting if necessary to a subsequence, we may assume that $u_m \rightharpoonup u = \partial_x g$ in X and replacing g_m by $g_m - g$, we may assume that $g = 0$. Denote by $F[u](r,s)$ the Fourier transform of $u(x,y)$.

Let

$$
\begin{aligned}
Q_{-1} &= \{(r,s) \in \mathbb{R}^n \mid |r| \le \rho,\ |s_i| \le \rho^2, i = 1,2,\cdots,n-1\},\\
Q_0 &= \{(r,s) \in \mathbb{R}^n \mid |r| > \rho\},\ Q_1 = \{(r,s) \in \mathbb{R}^n \mid |r| < \rho,\ |s_1| > \rho^2\},\\
&\vdots\\
Q_i &= \{(r,s) \in \mathbb{R}^n \mid |r| < \rho,\ |s_1| < \rho^2, \cdots,\ |s_{i-1}| < \rho^2,\ |s_i| > \rho^2\},\\
&\vdots\\
Q_{n-1} &= \{(r,s) \in \mathbb{R}^n \mid |r| < \rho,\ |s_1| < \rho^2, \cdots,\ |s_{n-2}| < \rho^2,\ |s_{n-1}| > \rho^2\}.
\end{aligned}
$$

Then $\mathbb{R}^n = \bigcup_{i=-1}^{n-1} Q_i$ and $Q_i \cap Q_j = \emptyset,\ i \ne j$.

For $\rho > 0$, there holds

$$
\int_{B(0,2R)} |u_m|^2\, dV = \int_{\mathbb{R}^n} |F[u_m]|^2\, drds = \sum_{i=-1}^{n-1} \int_{Q_i} \big|F[u_m]\big|^2\, drds. \tag{2.9}
$$

It is clear that

$$
\int_{Q_0} \big|F[u_m]\big|^2\, drds = \int_{Q_0} \frac{1}{4\pi^2 r^2} \big|F[\partial_x u_m]\big|^2\, drds \le \frac{1}{4\pi^2\rho^2} |\partial_x u_m|_2^2,
$$

and for $i = 1, \cdots, n-1$, there holds

$$
\int_{Q_i} \big|F[u_m]\big|^2\, dxdy = \int_{Q_i} \frac{r^2}{|s_i|^2} \big|F[v_{m,i}]\big|^2\, drds \le \frac{1}{\rho^2} |v_m|_2^2.
$$

For any $\varepsilon > 0$, there exists $\rho > 0$ large enough, such that

$$
\sum_{i=0}^{n-1} \int_{Q_i} \big|F[u_m]\big|^2\, drds \le \varepsilon/2.
$$

Since $u_m \rightharpoonup 0$ in $L^2(R^n)$, there holds

$$
F[u_m](r,s) = \int_{B(0,2R)} u_m(x,y) e^{-2i\pi(xr + y\cdot s)}\, dV \to 0, \text{ as } m \to \infty
$$

and

$$
\big|F[u_m](r,s)\big| \le c_0 |u_m|_2 \le c_1.
$$

Lebesgue's dominated convergence theorem implies that

$$\int_{Q_{-1}} \big|F[u_m]\big|^2 \, drds \to 0, \text{ as } m \to \infty.$$

Thus we have proved that $u_m \to 0$ in $L^2_{loc}(\mathbb{R}^n)$. By Lemma 2.3 and interpolation theorem, there holds $u_m \to 0$ in $L^p_{loc}(\mathbb{R}^n)$ if $2 \le p < \bar{p}$. ☑

Lemma 2.5. *If $\{u_m\}_{m=1}^{+\infty}$ is bounded in X and if*

$$\sup_{(x,y)\in R^n} \int_{B(x,y;r)} |u_m|^2 \, dV \to 0, \text{ as } n \to \infty. \tag{2.10}$$

Then $u_m \to 0$ in $L^p(\mathbb{R}^n)$ for $2 < p < \bar{p}$.

Proof. Let $2 < s < \bar{p}$ and $u \in X$. By Hölder inequality and Lemma 2.3, there holds

$$\begin{aligned} |u|_{L^s(B(x,y;r))} &\le |u|^{1-\lambda}_{L^2(B(x,y;r))} |u|^{\lambda}_{L^{\bar{p}}(B(x,y;r))} \\ &\le c_0 |u|^{1-\lambda}_{L^2(B(x,y;r))} \Big(\int_{B(x,y;r)} \big[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2\big] \, dV \Big)^{\frac{\lambda}{2}}, \end{aligned} \tag{2.11}$$

where $\frac{1}{s} = \frac{1-\lambda}{2} + \frac{\lambda}{\bar{p}}$. Choosing s such that $\frac{\lambda s}{2} = 1$, i.e., $s = \frac{2(2n+1)}{2n-1}$, there holds

$$\int_{B(x,y;r)} |u|^s \, dV \le c_0^s |u|^{(1-\lambda)s}_{L^2(B(x,y;r))} \int_{B(x,y;r)} \big[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2\big] \, dV. \tag{2.12}$$

Now, covering $\mathbb{R}^n$ by balls of radius r in such a way that each point of $\mathbb{R}^n$ is contained in at most 3 balls, then there holds

$$\int_{\mathbb{R}^n} |u|^s \, dV \le 3c_0^s \sup_{(x,y)\in R^n} |u|^{(1-\lambda)s}_{L^2(B(x,y;r))} \int_{\mathbb{R}^n} \big[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2\big] \, dV. \tag{2.13}$$

Under assumption (2.10), (2.13) implies $u_m \to 0$ in $L^s(\mathbb{R}^n)$. By Hölder inequality and Lemma 2.3, there holds $u_m \to 0$ in $L^p(\mathbb{R}^n)$ for all $2 < p < \bar{p}$. ☑

We recall the following Mountain Pass Lemma without (PS) condition as our Lemma 2.6 (cf. [1]).

Lemma 2.6 (Mountain Pass Lemma)**.** *Suppose X is a Banach space and $E \in C^1(X, R)$ satisfies the following geometrical properties:*

(1) *$E(0) = 0$, and there exists $\rho > 0$, such that $E\Big|_{\partial B_\rho(0)} \ge \alpha > 0$;*

(2) *There exists $e \in X \setminus \overline{B_\rho(0)}$, such that $E(e) \le 0$.*

Let Γ be the set of all passes which connects 0 and e, i.e.,

$$\Gamma = \{g \in C([0,1], E) \big| \; g(0) = 0, g(1) = e\}, \tag{2.14}$$

and

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} E(g(t)). \tag{2.15}$$

Then $c \geq \alpha$ and E possesses a $(PS)_c$ sequence at level c defined by (2.15), i.e., there exists a sequence $\{u_m\}_{m=1}^{+\infty}$ such that $E(u_m) \to c$ and $DE(u_m) \to 0$ as $m \to \infty$.

3. **Existence of nontrivial solitary waves**

The solitary waves of problem (1.1) satisfies:

$$\begin{cases} \big(- u_{xx} + D_x^{-2}\Delta_y u + cu - f(u)\big)_x = 0, \\ u \in X, \end{cases} \tag{3.1}$$

where $c > 0$. The weak solutions of (3.1) are the critical points of the functional E defined on X as

$$E(u) := \int_{\mathbb{R}^n} \big(\frac{1}{2}[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2] - F(u)\big)\, dV,$$

where $F(u) = \displaystyle\int_0^u f(s)\, ds$. Assume:

- (f_1) $f \in C^0(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and for some $2 < p < \bar{p} = \frac{2(2n-1)}{2n-3}$, $0 < c_0 < c$, $c_1 > 0$, there holds
$$|f(u)| \leq c_0|u| + c_1|u|^{p-1};$$
- (f_2) There exists $v \in X$ such that
$$\frac{f(\lambda v)}{\lambda} \to +\infty, \text{ as } \lambda \to +\infty;$$
- (f_3) There exists $\alpha > 2$ such that, for $u \in \mathbb{R}$, there holds
$$\alpha F(u) \leq uf(u).$$

By assumption (f_1) and Lemma 2.3, $E \in C^1(X, \mathbb{R})$.

Lemma 3.1. *Under assumptions (f_1) and (f_2), there exists $e \in X$ and $r > 0$ such that $\|e\| \geq r$ and*

$$b := \inf_{\|u\|=r} E(u) > E(0) = 0 \geq E(e).$$

Proof. From (f_1), there holds

$$|F(u)| = |\int_0^u f(s)\, ds| \leq c_0\frac{|u|^2}{2} + \frac{c_1}{p}|u|^p.$$

Then from the definition of the norm (2.2) in X, there holds

$$E(u) \geq \frac{\|u\|^2}{2} - \int_{R^n} \big(\frac{c_0}{2}|u|^2 + \frac{c_1}{p}|u|^p\big)\, dV \geq (\frac{1}{2} - \frac{c_0}{2c})\|u\|^2 - c_1|u|_p^p.$$

By Lemma 2.3, there exists $r > 0$ such that

$$b := \inf_{\|u\|=r} E(u) > E(0) = 0.$$

It follows from assumption (f_2) that

$$E(\lambda v) \to -\infty, \text{ as } \lambda \to +\infty.$$

Hence there exists $\lambda_0 > 0$ such that $e = \lambda_0 v$ satisfies $\|e\| > r,\ E(e) \leq 0.$ ☑

Define

$$d := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)),$$

$$\Gamma := \{\gamma \in C([0,1]; X) :\ \gamma(0) = 0,\ \gamma(1) = e\}.$$

Clearly, $d \geq b > 0$. Applying Lemma 2.6, there exists a $(PS)_c$ sequence $\{u_m\}_{m=1}^{+\infty}$ at level $c = d$ such that

$$E(u_m) \to d \text{ and } DE(u_m) \to 0 \text{ as } m \to \infty.$$

Theorem 3.2. *Under assumptions (f_1)–(f_3), problem* (3.1) *possesses a nontrivial solution.*

Proof. **1.** Boundness of $(PS)_c$ sequence.

Let $\{u_m\}_{m=1}^{+\infty}$ be the sequence derived by Lemma 2.6, i.e., $E(u_m) \to d$ and $DE(u_m) \to 0$ as $m \to \infty$. As $m \to \infty$, from assumption (f_3), there holds

$$\begin{aligned} d + o(1) + o(1)\|u_m\| \quad &\geq E(u_m) - \alpha^{-1}(DE(u_m), u_m) \\ &= (\tfrac{1}{2} - \tfrac{1}{\alpha})\|u_m\|^2 + \int_{\mathbb{R}^n} \left[\alpha^{-1}u_m f(u_m) - F(u_m)\right] dV \\ &\geq (\tfrac{1}{2} - \tfrac{1}{\alpha})\|u_m\|^2. \end{aligned}$$

Hence $\{u_m\}_{m=1}^{+\infty}$ is bounded in X.

2. $\delta := \overline{\lim_{m\to\infty}} \sup_{(x,y)\in R^n} \int_{B(x,y;\,1)} |u_m|^2\, dV \neq 0.$

Otherwise, by Lemma 2.5, there holds $u_m \to 0$ in $L^s(R^n)$ for $2 < s < \frac{2(2n-1)}{2n-3}$. It follows that

$$\begin{aligned} 0 < d \quad &= E(u_m) - \tfrac{1}{2}(DE(u_m), u_m) + o(1) \\ &= \int_{\mathbb{R}^n} [\frac{1}{2}u_m f(u_m) - F(u_m)]\, dV + o(1) = 0(1), \end{aligned}$$

which is a contradiction.

3. Existence of a nontrivial solution of problem (3.1).

Selecting if necessary a subsequence, we can assume that there existes a sequence $(x_m, y_m) \subset R^n$ such that

$$\int_{B(x_m,y_m;1)} |u_m|^2\, dV > \delta/2.$$

Define $v_m(x, y) := u_m(x + x_m, y + y_m)$ so that

$$\int_{B(0;1)} |v_m|^2\, dV > \delta/2.$$

Selecting if necessary a subsequence, we can assume that there existes a $v \in X$ such that

$$v_m \rightharpoonup v \text{ in } X, \text{ as } m \to \infty.$$

By Lemma 2.4, $v_m \to v$ in $L^2_{loc}(\mathbb{R}^n)$ and so $v \neq 0$, and for every $w \in X$, there holds

$$\int_{\mathbb{R}^n} (f(v_m) - f(v))w\, dV = \int_{B(0,R)} (f(v_m) - f(v))w\, dV + \int_{\mathbb{R}^n \setminus B(0,R)} (f(v_m) - f(v))w\, dV.$$

Since $w \in X$, then $w \in L^p(\mathbb{R}^n)$ and $\{v_m\}$ is bounded in X, hence $\{v_m\}$ is bounded in $L^p(\mathbb{R}^n)$, thus for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ large enough and independent on m such that

$$\int_{\mathbb{R}^n \setminus B(0,R)} (f(v_m) - f(v))w\, dV < \varepsilon, \ \forall m$$

On the other hand, for this $R > 0$, from Lemma 2.4, there holds

$$\int_{B(0,R)} (f(v_m) - f(v))w\, dV \to 0, \text{ as } m \to \infty.$$

Thus, there holds

$$\int_{\mathbb{R}^n} f(v_m)w\, dV \to \int_{\mathbb{R}^n} f(v)w\, dV, \text{ as } m \to \infty,$$

which implies

$$(DE(v), w) = \lim_{m\to\infty} (DE(v_m), w) = 0$$

Hence $DE(v) = 0$ and v is a nontrivial solution of problem (3.1). ☑

4. Nonexistence of nontrivial solitary waves

In this section, we derive a Pohozaev type variational identity of the solitary wave of problem:

$$\big(- u_{xx} + D_x^{-2}\Delta_y u - g(u)\big)_x = 0,$$

where $g \in C^1(\mathbb{R}, \mathbb{R})$ such that $g(0) = 0$ and define $G(u) := \int_0^u g(s)\, ds$.

First, we give a formal argument explaining the variational identity. For any $\lambda > 0$, define a transformation $T(\lambda) : X \to X$ as

$$T(\lambda)u(x, y) := u(x/\lambda, y/\lambda^2), \ (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Then $T(1) = \mathrm{id}_X$. If $u \in X$ is a critical point of functional $E(u)$, we conjecture that

$$\frac{\partial}{\partial \lambda}\Big|_{\lambda=1} E(T(\lambda)u) = 0. \tag{4.1}$$

A simple computation shows that

$$E(T(\lambda)u) = \frac{\lambda^{2n-3}}{2}\int_{\mathbb{R}^n} \big(u_x^2 + |D_x^{-1}\nabla_y u|^2\big)\,dV - \lambda^{2n-1}\int_{\mathbb{R}^n} G(u)\,dV. \tag{4.2}$$

and

$$\begin{aligned}&\frac{\partial}{\partial\lambda}\Big|_{\lambda=1} E(T(\lambda)u) = \\ &\qquad \frac{2n-3}{2}\int_{\mathbb{R}^n} \big(u_x^2 + |D_x^{-1}\nabla_y u|^2\big)\,dV - (2n-1)\int_{\mathbb{R}^n} G(u)\,dV,\end{aligned} \tag{4.3}$$

which implies that

$$\int_{\mathbb{R}^n} \big(u_x^2 + |D_x^{-1}\nabla_y u|^2\big)\,dV = \frac{2(2n-1)}{2n-3}\int_{\mathbb{R}^n} G(u)\,dV. \tag{4.4}$$

In fact, we have the following Theorem:

Theorem 4.1. *Any solution of*

$$\begin{cases} \big(-u_{xx} + D_x^{-2}\Delta_y u - g(u)\big)_x = 0, \\ u \in X \cap H^2_{loc}(\mathbb{R}^n), \\ G(u), g(u)u \in L^1(\mathbb{R}^n),\ g(u)D_x^{-1}\nabla_y u \in (L^1(\mathbb{R}^n))^{n-1}, \end{cases} \tag{4.5}$$

satisfies (4.4).

Proof. **1.** Let

$$J(u) := \int_{\mathbb{R}^n} \big(\frac{1}{2}[u_x^2 + |D_x^{-1}\nabla_y u|^2] - G(u)\big)\,dV.$$

Then a weak solution of problem (4.5) is a critical point of operator J. Let $\psi \in \mathcal{D}(\mathbb{R})$ be such that $0 \le \psi \le 1, \psi(r) = 1$ for $r = 1$ and $\psi(r) = 0$ for $r \ge 2,\ |\psi'(r)| \le 2,\ |\psi''(r)| \le 4$. Define a sequence of functions on $\mathbb{R}^n$ as:

$$\psi_m(x,y) := \psi\Big(\frac{x^2+|y|^2}{m^2}\Big),\ \forall (x,y) \in \mathbb{R}^n.$$

2. For any solution of problem (4.5), there holds

$$\frac{3}{2}\int_{\mathbb{R}^n} u_x^2\,dV - \frac{1}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\,dV + \int_{\mathbb{R}^n} \big(G(u) - g(u)u\big)\,dV = 0. \tag{4.6}$$

For every integer m, there holds

$$\int_{\mathbb{R}^n} \big(-u_{xx} + D_x^{-2}\Delta_y u - g(u)\big)\big(\psi_m x u\big)_x\,dV = 0. \tag{4.7}$$

Integrating by parts, there holds

$$-\int\limits_{\mathbb{R}^n} u_{xx}\big(\psi_m xu\big)_x\,dV = -\int\limits_{\mathbb{R}^n} u_{xx}\big(\psi_{m,x}xu + \psi_m u + \psi_m xu_x\big)\,dV$$
$$= \int\limits_{\mathbb{R}^n} \left[\frac{3}{2}u_x^2(\psi_{m,x}x + \psi_m) + 2\psi_{m,x}uu_x + \psi_{m,xx}xuu_x\right]\,dV.$$

Lebesgue dominated convergence theorem implies that, as $m \to \infty$, there holds

$$-\int_{\mathbb{R}^n} u_{xx}\big(\psi_m xu\big)_x\,dV = \frac{3}{2}\int_{\mathbb{R}^n} u_x^2\,dV + o(1). \tag{4.8}$$

Similarly, there hold

$$\begin{aligned}
&\int_{\mathbb{R}^n} D_x^{-2}\Delta_y u\big(\psi_m xu\big)_x\,dV \\
&\quad= -\int_{\mathbb{R}^n} \big(D_x^{-1}\Delta_y u\big)\big(\psi_m xu\big)\,dV \\
&\quad= -\int_{\mathbb{R}^n} \sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}\big(D_x^{-1}u_{y_i}\big)\big(\psi_m xu\big)\,dV \\
&\quad= \int_{\mathbb{R}^n} \sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\frac{\partial}{\partial y_i}\big(\psi_m xu\big)\,dV \\
&\quad= \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\psi_{m,y_i}xu + \sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\psi_m x\frac{\partial}{\partial x}D_x^{-1}u_{y_i}\Big)\,dV \\
&\quad= \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{n-1} D_x^{-1}u_{y_i}\psi_{m,y_i}xu - \frac{1}{2}\sum_{i=1}^{n-1}|D_x^{-1}u_{y_i}|^2(\psi_{m,x}x + \psi_m)\Big)\,dV \\
&\quad= -\frac{1}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\,dV + o(1),
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&-\int_{\mathbb{R}^n} g(u)\big(\psi_m xu\big)_x\,dV \\
&\quad= -\int_{\mathbb{R}^n} g(u)\big(\psi_{m,x}xu + \psi_m u + \psi_m xu_x\big)\,dV \\
&\quad= -\int_{\mathbb{R}^n} \big(g(u)\psi_m u + g(u)\psi_{m,x}xu + \frac{dG(u)}{dx}\psi_m x\big)\,dV \\
&\quad= \int_{\mathbb{R}^n} (G(u) - g(u)u)\,dV + o(1).
\end{aligned} \tag{4.10}$$

Substituting (4.8)–(4.10) into (4.7) yields (4.6)

3. On the other hand, since u is a weak solution of problem (4.5), i.e., $DJ(u) = 0$, then from $(DJ(u), u) = 0$, there holds

$$\int_{\mathbb{R}^n} \big(u_x^2 + |D_x^{-1}\nabla_y u|^2\big)\, dV = \int_{\mathbb{R}^n} g(u)u\, dV. \tag{4.11}$$

4. For any solution of problem (4.5), there holds

$$-\frac{n-1}{2}\int_{\mathbb{R}^n} u_x^2\, dV - \frac{n-3}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV + (n-1)\int_{\mathbb{R}^n} G(u)\, dV = 0. \tag{4.12}$$

For every integer m, there also holds

$$\int_{\mathbb{R}^n} \big(-u_{xx} + D_x^{-2}\Delta_y u - g(u)\big)\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)_x\, dV = 0. \tag{4.13}$$

Integrating by parts and applying Lebesgue dominated convergence theorem imply that, as $m\to\infty$, there hold

$$\begin{aligned}
&-\int_{\mathbb{R}^n} u_{xx}\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)_x\, dV\\
&\quad= -\int_{\mathbb{R}^n} u_{xx}\big(\psi_{m,x} y\cdot D_x^{-1}\nabla_y u + \psi_m y\cdot\nabla_y u\big)\, dV\\
&\quad= \int_{\mathbb{R}^n} u_x\big(\psi_{m,x} y\cdot D_x^{-1}\nabla_y u + \psi_m y\cdot\nabla_y u\big)_x\, dV\\
&\quad= \int_{\mathbb{R}^n} u_x\big(\psi_{m,xx} y\cdot D_x^{-1}\nabla_y u + 2\psi_{m,x} y\cdot\nabla_y u + \psi_m y\cdot\nabla_y u_x\big)\, dV\\
&\quad= -\frac{n-1}{2}\int_{\mathbb{R}^n} u_x^2\, dV + o(1),
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big(D_x^{-2}\Delta_y u\big)\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)_x\, dV\\
&\quad= -\int_{\mathbb{R}^n} \big(D_x^{-1}\Delta_y u\big)\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)\, dV\\
&\quad= -\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{n-1}\frac{\partial}{\partial y_i}(D_x^{-1}u_{y_i})\Big)\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)\, dV\\
&\quad= \int_{\mathbb{R}^n}\sum_{i=1}^{n-1}(D_x^{-1}u_{y_i})\big(\psi_m y\cdot D_x^{-1}\nabla_y u\big)_{y_i}\, dV\\
&\quad= -\frac{n-3}{2}\int_{\mathbb{R}^n} |D_x^{-1}\nabla_y u|^2\, dV + o(1)
\end{aligned} \tag{4.15}$$

and

$$
\begin{aligned}
&-\int_{\mathbb{R}^n} g(u)\big(\psi_m y \cdot D_x^{-1}\nabla_y u\big)_x \, dV \\
&\quad = -\int_{\mathbb{R}^n} g(u)\big(\psi_{m,x} y \cdot D_x^{-1}\nabla_y u + \psi_m y \cdot \nabla_y u\big)\, dV \\
&\quad = -\int_{\mathbb{R}^n} \Big(g(u)\psi_{m,x} y \cdot D_x^{-1}\nabla_y u + \sum_{i=1}^{n-1} \frac{dG(u)}{dy_i} y_i \psi_m\Big)\, dV \\
&\quad = (n-1)\int_{\mathbb{R}^n} G(u)\, dV + o(1).
\end{aligned}
\tag{4.16}
$$

Thus, from equations (4.13)–(4.16) (4.12) holds. Equations (4.6), (4.11) and (4.12) imply equation (4.4). ☑

Theorem 4.2. *(Nonexistence of nontrivial solitary wave) If $g \in C^1(\mathbb{R};\mathbb{R})$ satisfies $g(0) = 0$ and*

$$
\frac{2(2n-1)}{2n-3} G(u) - g(u)u < 0, \ \forall u \neq 0, \tag{4.17}
$$

then 0 is the only solution of problem (4.5).

Proof. If $u \not\equiv 0$ is a solution of problem (4.5), then (4.4)-(4.11), there holds

$$
\int_{\mathbb{R}^n} \big[\frac{2(2n-1)}{2n-3} G(u) - g(u)u\big]\, dV = 0
$$

which contradicts (4.17). ☑

Corollary 4.3. *Let $c > 0$, and $p \geq \frac{2(2n-1)}{2n-3}$, then 0 is the only solution of problem:*

$$
\begin{cases}
\big(-u_{xx} + D_x^{-2}\Delta_y u + cu - |u|^{p-2}u\big)_x = 0, \\
u \in X \cap H^2_{loc}(\mathbb{R}^n), \\
|u|^{p-2} u D_x^{-1}\nabla_y u \in (L^1(\mathbb{R}^n))^{n-1}.
\end{cases}
\tag{4.18}
$$

Proof. Since $g(u) = |u|^{p-2}u - cu$, then $G(u) = \dfrac{1}{p}|u|^p - \dfrac{c}{2}u^2$, thus (4.17) holds. ☑

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